

Regular approximations of spectra of singular discrete linear Hamiltonian systems with one singular endpoint*

Yan Liu, Yuming Shi **

School of Mathematics, Shandong University

Jinan, Shandong 250100, P. R. China

Abstract. This paper is concerned with regular approximations of spectra of singular discrete linear Hamiltonian systems with one singular endpoint. For any given self-adjoint subspace extension (SSE) of the corresponding minimal subspace, its spectrum can be approximated by eigenvalues of a sequence of induced regular SSEs, generated by the same difference expression on smaller finite intervals. It is shown that every SSE of the minimal subspace has a pure discrete spectrum, and the k -th eigenvalue of any given SSE is exactly the limit of the k -th eigenvalues of the induced regular SSEs; that is, spectral exactness holds, in the limit circle case. Furthermore, error estimates for the approximations of eigenvalues are given in this case. In addition, in the limit point and intermediate cases, spectral inclusive holds.

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1 Introduction

Consider the following discrete linear Hamiltonian system:

$$J\Delta y(t) = (P(t) + \lambda W(t))R(y)(t), \quad t \in \mathcal{I}, \quad (1.1_\lambda)$$

where $\mathcal{I} := \{t\}_{t=a}^{+\infty}$ is an integer interval, a is an integer; J is the $2n \times 2n$ canonical symplectic matrix, i.e.,

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

with the $n \times n$ identity matrix I_n ; Δ is the forward difference operator, i.e., $\Delta y(t) = y(t+1) - y(t)$; the weight function $W(t) = \text{diag}\{W_1(t), W_2(t)\}$, $W_1(t)$ and $W_2(t)$ are $n \times n$ positive

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** The corresponding author.

Email addresses: yanliumaths@126.com (Y. Liu), ymshi@sdu.edu.cn (Y. Shi)

semi-definite matrices; $P(t)$ is a $2n \times 2n$ Hermitian matrix; the partial right shift operator $R(y)(t) = (y_1^T(t+1), y_2^T(t))^T$ with $y(t) = (y_1^T(t), y_2^T(t))^T$ and $y_1(t), y_2(t) \in \mathbf{C}^n$; λ is a complex spectral parameter.

It is evident that $P(t)$ can be blocked as

$$P(t) = \begin{pmatrix} -C(t) & A^*(t) \\ A(t) & B(t) \end{pmatrix},$$

where $A(t)$, $B(t)$, and $C(t)$ are $n \times n$ complex-valued matrices, $B(t)$ and $C(t)$ are Hermitian matrices, and $A^*(t)$ is the complex conjugate transpose of $A(t)$. Then system (1.1) $_\lambda$ can be written as

$$\begin{aligned} \Delta y_1(t) &= A(t)y_1(t+1) + (B(t) + \lambda W_2(t))y_2(t), \\ \Delta y_2(t) &= (C(t) - \lambda W_1(t))y_1(t+1) - A^*(t)y_2(t), \quad t \in \mathcal{I}. \end{aligned} \quad (1.2)$$

To ensure the existence and uniqueness of the solution of any initial value problem for (1.1) $_\lambda$, we always assume that

(A₁) $I_n - A(t)$ is invertible in \mathcal{I} .

It is known that (1.1) $_\lambda$ contains the following formally self-adjoint vector difference equation of order $2m$:

$$\sum_{j=0}^m (-1)^j \Delta^j [p_j(t) \Delta^j z(t-j)] = \lambda w(t) z(t), \quad t \in \mathcal{I}, \quad (1.3)$$

where $w(t)$ and $p_j(t)$, $0 \leq j \leq m$, are $l \times l$ Hermitian matrices, $w(t) \geq 0$, and $p_m(t)$ is invertible in \mathcal{I} . The reader is referred to [28] for the details.

Spectral problems can be divided into two classifications. Those defined over finite closed intervals with well-behaved coefficients are called regular; otherwise they are called singular.

With the development of information technology and the wide applications of digital compute, more and more discrete systems have appeared and they have attracted a lot of attention. The study of fundamental theory of regular difference equations has a long history and their spectral theory has formed a relatively complete theoretical system such as eigenvalue problems, orthogonality of eigenfunctions and expansion theory (cf., [2, 17, 27, 36, 39, 41]). Spectral problems for singular difference equations were firstly studied by Atkinson [2] in 1964, and some significant progresses have been made since then (cf., e.g., [5, 6, 8, 16, 21, 22, 24, 25, 28, 32, 33, 37, 38]). Especially, research on spectral theory of singular discrete Hamiltonian systems has attracted a great deal of interest and some good results have been obtained (cf., [22, 24, 25, 28, 37, 38], and references cited therein). In 2006, the second author of the present paper established the Weyl-Titchmarsh theory for system (1.1) with $a = 0$ in [28]. Later, she with Ren studied the defect indices and definiteness conditions and gave out complete characterizations of self-adjoint extensions for system (1.1) [24, 25].

Recently, she with Sun studied some spectral properties of system (1.1) [37]. These results have laid a foundation of our present research.

It is well known that regular discrete spectral problems have finite and then discrete spectra. In particular, they can be transformed into eigenvalue problems of a special kind of matrices. So they can be easily calculated by computer. Compared with regular problems, the spectral set of a singular discrete spectral problem may contain some essential spectral points except for isolated spectral points. Thus, it is difficult to study them. It is interesting to ask whether the spectra of a singular spectral problem can be approximated by those of regular spectral problems, and how to do it. Obviously, the study of regular approximations of spectra of singular spectral problems plays an important role in both theory and practical applications.

Regular approximations of spectra of singular differential equations have been investigated widely and deeply, and some good results have been obtained, including spectral inclusion and spectral exactness [3, 4, 7, 18, 34, 35, 40, 43, 44].

To the best of our knowledge, there seem a few results about regular approximations of spectra of singular difference equations. Recently, we studied this problem for singular second-order symmetric linear difference equations [19, 20]. For each self-adjoint subspace extension of a given singular second-order symmetric linear difference equation, we constructed a sequence of regular problems and showed that the spectrum of the singular problem can be approximated by the eigenvalues of this sequence. Motivated by the ideas and methods used in [19, 20], we shall study similar problems for singular discrete Hamiltonian system $(1.1)_\lambda$ in the present paper. Although the methods are similar to that used in [19, 20], the problems investigated in the present paper are more complicated and difficult. This results from the higher dimension and the partial shift operator R in system $(1.1)_\lambda$. We shall point out that there is another difficulty that will not be encountered in the continuous case. It is that the maximal operator generated by (1.1) may be multi-valued, and the corresponding minimal operator may be multi-valued or non-densely defined (see the detailed discussions in [24, 25, 29, 32]). These facts were ignored in some existing literature including [28]. This is an essential difficulty that one would encounter in the study of the regular approximations of spectra for difference expressions because the corresponding theory of linear operators is not applicable in this case.

Fortunately, this major difficulty can be overcome by using the theory of linear subspaces (i.e., linear relations). In 1961, Arens [1] initiated the study of linear relations, and his work was followed by many scholars [9-15]. Recently, some fundamental results of Hermitian subspaces including the Glazman-Krein-Naimark theory, fundamental spectral properties of self-adjoint subspaces, and the resolvent convergence and spectral approximations of sequences of self-adjoint subspaces were established [29-31]. A linear relation is actually a

subspace in a related product space, and obviously includes multi-valued and non-densely defined linear operators in the related space. Therefore, we shall study the regular approximations of spectra of system (1.1) in the framework of subspaces in a product space.

The rest of this paper is organized as follows. In Section 2, some basic concepts and fundamental results about subspaces and system (1.1) are introduced, including the maximal and minimal subspaces for (1.1), spectral inclusion, and spectral exactness. In Section 3, the induced regular SSEs for any given SSE are constructed. Section 4 pays attention to how to extend a subspace in the product space of the fundamental spaces on a proper subinterval to a subspace in that on the original interval, i.e., how to do the “zero extensions”. This problem can be very easily solved in the continuous case but hard in the discrete case. Further, the invariance of spectral properties of the extended subspaces is given. As a consequence, the extension from the induced regular SSE to a subspace in the product space of the original Hilbert spaces is given, and the invariance of spectral properties of the extended subspaces is obtained. Regular approximations of spectra of system (1.1) in the limit circle case are studied in Section 5. It is shown that the sequence of induced regular SSEs constructed in Section 3 is spectrally exact for any given SSE in this case. In addition, it is obtained that the k -th eigenvalue of any given SSE is exactly the limit of the k -th eigenvalues of the induced regular SSEs in this case. Furthermore, error estimates for the approximations of eigenvalues are given in this case. Section 6 is concerned with regular approximations of spectra of system (1.1) in the limit point and intermediate cases. It is only shown that spectral inclusion holds in each case.

Remark 1.1. We shall further study regular approximations of spectra of singular discrete linear Hamiltonian systems with two singular endpoints in our forthcoming paper.

2 Preliminaries

This section is divided into three parts. In Section 2.1, we recall some basic concepts and fundamental results about subspaces. In Section 2.2, we first introduce the maximal, pre-minimal, and minimal subspaces corresponding to (1.1). Then, we list some useful results about (1.1), which will be used in the sequent sections. Some useful results about resolvent convergence of sequences of self-adjoint subspaces are introduced in Section 2.3.

2.1 Some basic concepts and fundamental results about subspaces

By \mathbf{C} , \mathbf{R} and \mathbf{Z}^+ denote the sets of the complex numbers, real numbers, and positive integer numbers, respectively. Let X be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and T a

linear subspace (briefly, subspace) in the product space X^2 with the following induced inner product, still denoted by $\langle \cdot, \cdot \rangle$ without any confusion:

$$\langle (x, f), (y, g) \rangle = \langle x, y \rangle + \langle f, g \rangle, \quad (x, f), (y, g) \in X^2.$$

The domain $D(T)$, range $R(T)$, and null space $N(T)$ of T are respectively defined by

$$\begin{aligned} D(T) : &= \{x \in X : (x, f) \in T \text{ for some } f \in X\}, \\ R(T) : &= \{f \in X : (x, f) \in T \text{ for some } x \in X\}, \\ N(T) : &= \{x \in X : (x, 0) \in T\}. \end{aligned}$$

Its adjoint subspace T^* is defined by

$$T^* = \{(y, g) \in X^2 : \langle f, y \rangle = \langle x, g \rangle \text{ for all } (x, f) \in T\}.$$

Further, denote

$$T(x) := \{f \in X : (x, f) \in T\}, \quad T^{-1} := \{(f, x) : (x, f) \in T\}.$$

It is evident that $T(0) = \{0\}$ if and only if T can uniquely determine a single-valued linear operator from $D(T)$ into X whose graph is T . A single-valued linear operator is briefly called a linear operator. For convenience, a linear operator in X will always be identified with a subspace in X^2 via its graph.

A subspace $T \subset X^2$ is called a Hermitian subspace if $T \subset T^*$, and it is called a self-adjoint subspace if $T = T^*$. A Hermitian subspace S is called a Hermitian subspace extension of T if $T \subset S$, and it is called a self-adjoint subspace extension of T if $T \subset S$ and S is a self-adjoint subspace. In addition, a subspace T is a Hermitian subspace if and only if $\langle f, y \rangle = \langle x, g \rangle$ for all $(x, f), (y, g) \in T$.

Let T and S be two subspaces in X^2 and $\alpha \in \mathbf{C}$. Define

$$\begin{aligned} \alpha T &:= \{(x, \alpha f) : (x, f) \in T\}, \\ T + S &:= \{(x, f + g) : (x, f) \in T, (x, g) \in S\}, \\ ST &:= \{(x, g) \in X^2 : (x, f) \in T, (f, g) \in S \text{ for some } f \in X\}. \end{aligned}$$

It is evident that if T is closed, then $T - \lambda I_{id}$ is closed and $(T - \lambda I_{id})^* = T^* - \bar{\lambda} I_{id}$, where $I_{id} := \{(x, x) : x \in X\}$, without any confusion we briefly denote it by I .

For the following definition, the reader is referred to [15, 30, 31].

Definition 2.1. Let T be a subspace in X^2 .

- (1) The set $\rho(T) := \{\lambda \in \mathbf{C} : (\lambda I - T)^{-1} \text{ is a bounded linear operator defined on } X\}$ is called the resolvent set of T .
- (2) The set $\sigma(T) := \mathbf{C} \setminus \rho(T)$ is called the spectrum of T .

Lemma 2.1 [30, Lemma 2.1]. *Let T be a closed subspace in X^2 . Then*

$$\begin{aligned}\rho(T^{-1}) \setminus \{0\} &= \{\lambda^{-1} : \lambda \in \rho(T) \text{ with } \lambda \neq 0\}, \\ \sigma(T^{-1}) \setminus \{0\} &= \{\lambda^{-1} : \lambda \in \sigma(T) \text{ with } \lambda \neq 0\}.\end{aligned}$$

Consequently, if $\rho(T) \neq \emptyset$, then

$$\sigma((\lambda_0 I - T)^{-1}) \setminus \{0\} = \{(\lambda_0 - \lambda)^{-1} : \lambda \in \sigma(T)\}, \quad \lambda_0 \in \rho(T).$$

Lemma 2.2 [19, Lemma 2.1]. *Let T be a closed subspace in X^2 . Then $\lambda \in \rho(T)$ if and only if $R(\lambda I - T) = X$ and $N(\lambda I - T) = \{0\}$.*

Lemma 2.3 [30, Theorem 3.6]. *Assume that X_1 is a proper closed subspace in X , $P : X \rightarrow X_1$ the orthogonal projection, and T a self-adjoint subspace in X_1^2 . Then*

- (i) $T' = TG(P)$ is a self-adjoint subspace in X^2 with $D(T') = D(T) \oplus X_1^\perp$;
- (ii) $\sigma(T') = \sigma(T) \cup \{0\}$.

2.2 Maximal, pre-minimal, and minimal subspaces

In this subsection, we first introduce the concepts of maximal, pre-minimal, and minimal subspaces, and then list some useful results about system (1.1 _{λ}).

For any integer interval $\mathcal{I} = \{t\}_{t=a}^b$ with $-\infty < a < b \leq +\infty$, we denote

$$\mathcal{I}^+ := \{t\}_{t=a}^{b+1}, \quad l(\mathcal{I}) := \{y : y = \{y(t)\}_{\mathcal{I}^+} \subset \mathbf{C}^{2n}\},$$

where $b+1$ means $+\infty$ in the case of $b = +\infty$. Denote

$$\mathcal{L}_W^2(\mathcal{I}) := \left\{ y \in l(\mathcal{I}) : \sum_{t \in \mathcal{I}} R(y)^*(t)W(t)R(y)(t) < +\infty \right\}$$

with the semi-scalar product

$$\langle x, y \rangle := \sum_{t \in \mathcal{I}} R^*(y)(t)W(t)R(x)(t).$$

Further, we define $\|y\| := (\langle y, y \rangle)^{1/2}$ for $y \in \mathcal{L}_W^2(\mathcal{I})$. Since the weighted function $W(t)$ may be singular in \mathcal{I} , $\|\cdot\|$ is a semi-norm. We denote

$$L_W^2(\mathcal{I}) := \mathcal{L}_W^2(\mathcal{I}) / \{y \in \mathcal{L}_W^2(\mathcal{I}) : \|y\| = 0\}.$$

Then $L_W^2(\mathcal{I})$ is a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ (cf. [28, Lemma 2.5]). For a function $y \in \mathcal{L}_W^2(\mathcal{I})$, we denote by \tilde{y} the corresponding equivalent class in $L_W^2(\mathcal{I})$. And for any $\tilde{y} \in L_W^2(\mathcal{I})$, by $y \in \mathcal{L}_W^2(\mathcal{I})$ denote a representative of \tilde{y} . It is evident that $\langle \tilde{x}, \tilde{y} \rangle = \langle x, y \rangle$ for any $\tilde{x}, \tilde{y} \in L_W^2(\mathcal{I})$. Set

$$\mathcal{L}_{W,0}^2(\mathcal{I}) := \{y \in \mathcal{L}_W^2(\mathcal{I}) : \text{there exist two integer } s, k \in \mathcal{I} \text{ with } s \leq k \text{ such that } y(t) = 0 \text{ for } t \leq s \text{ and } t \geq k+1\}.$$

The natural difference operator corresponding to system (1.1) $_{\lambda}$ is

$$\mathcal{L}(y)(t) := J\Delta y(t) - P(t)R(y)(t).$$

Set

$$H := \{(\tilde{y}, \tilde{g}) \in (L_W^2(\mathcal{I}))^2 : \text{there exists } y \in \tilde{y} \text{ such that } \mathcal{L}(y)(t) = W(t)R(g)(t), t \in \mathcal{I}\},$$

$$H_{00} := \{(\tilde{y}, \tilde{g}) \in H : \text{there exists } y \in \tilde{y} \text{ such that } y \in \mathcal{L}_{W,0}^2(\mathcal{I}) \text{ and } \mathcal{L}(y)(t) = W(t)R(g)(t), t \in \mathcal{I}\},$$

$$H_0 := \overline{H_{00}},$$

where H , H_{00} , and H_0 are called the maximal, pre-minimal, and minimal subspaces corresponding to system (1.1), respectively. By [24, Theorem 3.1], $H_{00}^* = H_0^* = H$, which implies that H_0 is a closed Hermitian subspace in $(L_W^2(\mathcal{I}))^2$.

By n_{λ} denote the number of linearly independent square summable solutions of (1.1) $_{\lambda}$ in $\mathcal{L}_W^2(\mathcal{I})$, and by d_{λ} denote the defect index of H_0 and $\bar{\lambda}$. By [24, Corollary 5.1] we know that $n_{\lambda} = d_{\lambda}$ if and only if the following definiteness condition is satisfied:

(A₂) There exists a finite subset $\mathcal{I}_0 := [s_0, t_0] \subset \mathcal{I}$ such that for some $\lambda \in \mathbf{C}$, any non-trivial solution $y(t)$ of (1.1) $_{\lambda}$ satisfies

$$\sum_{t \in \mathcal{I}_0} R(y)^*(t)W(t)R(y)(t) > 0.$$

Remark 2.1.

- (1) It has been shown in [24] that if the inequality in (A₂) holds for some $\lambda \in \mathbf{C}$, then it holds for all $\lambda \in \mathbf{C}$. Several sufficient conditions for (A₂) were given in [24]. Furthermore, it was pointed out that H_0 may be non-densely defined or multi-valued in [25, Section 6].
- (2) It has been shown that (A₂) is equivalent to that for any $(\tilde{y}, \tilde{g}) \in H$, there exists a unique $y \in \tilde{y}$ such that $\mathcal{L}(y)(t) = W(t)R(g)(t)$ for $t \in \mathcal{I}$ in [24, Theorem 4.2]. In this case, we briefly write $(y, \tilde{g}) \in H$ in the rest of the paper.
- (3) Even if (A₁) and (A₂) hold, $L_W^2(\mathcal{I})$ may be finite-dimensional since $W(t) \geq 0$ for $t \in \mathcal{I}$. In this case, $d_{\lambda} \equiv 2n$ for any $\lambda \in \mathbf{C}$.

In the sequel, it is always assumed that (A₂) holds. It has been shown by [28, Corollary 4.1] that $n_{\lambda} \geq n$ for each $\lambda \in \mathbf{C} \setminus \mathbf{R}$. Let d_{\pm} be the positive and negative indices of H_0 . Since $n_{\lambda} \leq 2n$ and $n_{\lambda} = d_{\lambda}$ for each $\lambda \in \mathbf{C}$, we have $n \leq d_{\pm} \leq 2n$. By [9, Corollary of Theorem 15 and Theorem 18], H_0 has an SSE in $L_W^2(\mathcal{I})$ if and only if $d_+ = d_-$. So we always assume that the following holds in the sequel:

(**A**₃) $d_+ = d_- =: d$.

In the minimal deficiency case of $d = n$, \mathcal{L} is said to be in the limit point case (l.p.c.) at $t = +\infty$ and in the maximal deficiency case of $d = 2n$, \mathcal{L} is said to be in the limit circle case (l.c.c.) at $t = +\infty$. We refer to the cases when $n < d < 2n$ as \mathcal{L} in the intermediate cases.

Next, for any $x, y \in l(\mathcal{I})$, we denote

$$(x, y)(t) = y^*(t)Jx(t).$$

In the case of $b = +\infty$, if $\lim_{t \rightarrow b}(x, y)(t)$ exists and is finite, then its limit is denoted by $(x, y)(+\infty)$.

By [28, Lemma 2.1], one has that for any $x, y \in l(\mathcal{I})$ and any $s, k \in \mathcal{I}$,

$$\sum_{t=s}^k [R(y)^*(t)\mathcal{L}(x)(t) - \mathcal{L}(y)^*(t)R(x)(t)] = (x, y)(t)|_s^{k+1}. \quad (2.1)$$

Hence, for $(x, \tilde{f}), (y, \tilde{g}) \in H$, we get from (2.1) that

$$\begin{aligned} & \sum_{t=s}^k [R(y)^*(t)W(t)R(f)(t) - R(g)^*(t)W(t)R(x)(t)] \\ &= \sum_{t=s}^k [R(y)^*(t)\mathcal{L}(x)(t) - \mathcal{L}(y)^*(t)R(x)(t)] \\ &= (x, y)(t)|_s^{k+1}, \end{aligned}$$

which yields that $\lim_{t \rightarrow +\infty}(x, y)(t)$ exists and is finite for all $(x, \tilde{f}), (y, \tilde{g}) \in H$. Further, by [28, Theorem 2.1] we get that for any $\lambda \in \mathbf{C}$, $c_0 \in \mathcal{I}$, and any solutions $y_\lambda(t)$ and $y_{\bar{\lambda}}(t)$ of (1.1 _{λ}) and (1.1 _{$\bar{\lambda}$}), respectively,

$$(y_\lambda, y_{\bar{\lambda}})(t) = (y_\lambda, y_{\bar{\lambda}})(c_0), \quad t \in \mathcal{I}^+. \quad (2.2)$$

Lemma 2.4 [25, Lemma 3.3]. *Assume that (**A**₁) and (**A**₂) hold. Then for any given finite subset $\tilde{\mathcal{I}} = \{t\}_{t=s}^k$ with $\mathcal{I}_0 \subset \tilde{\mathcal{I}} \subset \mathcal{I}$ and for any given $\alpha, \beta \in \mathbf{C}^{2n}$, there exists $g = \{g(t)\}_{t=s}^{k+1} \in \mathbf{C}^{2n}$ such that the following boundary value problem:*

$$\begin{aligned} & \mathcal{L}(y)(t) = W(t)R(g)(t), \quad t \in \tilde{\mathcal{I}}, \\ & y(s) = \alpha, \quad y(k+1) = \beta, \end{aligned}$$

has a solution $y = \{y(t)\}_{t=s}^{k+1} \in \mathbf{C}^{2n}$.

The following four lemmas are about SSE of H_0 and will be used in constructing proper induced regular SSEs for any given SSE of H_0 .

Lemma 2.5 [25, Theorem 5.12]. *Assume that (**A**₁) and (**A**₂) hold and $\mathcal{I} = \{t\}_{t=a}^b$ is finite. Then a subspace $H_1 \subset (L_W^2(\mathcal{I}))^2$ is an SSE of H_0 if and only if there exist two $2n \times 2n$ matrices M and N such that*

$$\begin{aligned} \text{rank}(M, N) &= 2n, & MJM^* &= NJN^*, \\ H_1 &= \{(y, \tilde{g}) \in H : My(a) - Ny(b+1) = 0\}. \end{aligned} \quad (2.3)$$

Lemma 2.6 [25, Theorem 5.10]. Assume that (\mathbf{A}_1) , (\mathbf{A}_2) , and (\mathbf{A}_3) hold and \mathcal{L} is in l.c.c. at $t = +\infty$. Let $\theta_1, \theta_2, \dots, \theta_{2n}$ be $2n$ linearly independent solutions of (1.1 $_\lambda$) with $\lambda \in \mathbf{R}$ and satisfy the following initial condition:

$$(\theta_1, \theta_2, \dots, \theta_{2n})(a, \lambda) = I_{2n}. \quad (2.4)$$

Then a subspace $H_1 \subset (L_W^2(\mathcal{I}))^2$ is an SSE of H_0 if and only if there exist two $2n \times 2n$ matrices M and N such that

$$\text{rank}(M, N) = 2n, \quad MJM^* = NJN^*, \quad (2.5)$$

$$H_1 = \{(y, \tilde{g}) \in H : My(a) - N \begin{pmatrix} (y, \theta_1)(+\infty) \\ \vdots \\ (y, \theta_{2n})(+\infty) \end{pmatrix} = 0\}. \quad (2.6)$$

Lemma 2.7 [25, Theorem 5.9]. Assume that (\mathbf{A}_1) , (\mathbf{A}_2) , and (\mathbf{A}_3) hold and \mathcal{L} is in l.p.c. at $t = +\infty$. Then a subspace $H_1 \subset (L_W^2(\mathcal{I}))^2$ is an SSE of H_0 if and only if there exists a matrix $M_{n \times 2n}$ satisfying the self-adjoint conditions:

$$\text{rank } M = n, \quad MJM^* = 0 \quad (2.7)$$

such that H_1 can be defined by

$$H_1 = \{(y, \tilde{g}) \in H : My(a) = 0\}. \quad (2.8)$$

Lemma 2.8 [25, Theorem 5.8]. Assume that (\mathbf{A}_1) , (\mathbf{A}_2) , and (\mathbf{A}_3) hold and \mathcal{L} is in the intermediate case at $t = +\infty$; that is, $n < d < 2n$. And assume that there exists $\lambda_0 \in \mathbf{R}$ such that system (1.1) has d linear independent solutions ψ_1, \dots, ψ_d in $\mathcal{L}_W^2(\mathcal{I})$. Let them be arranged such that

$$\Lambda := ((\psi_i, \psi_j)(+\infty))_{1 \leq i, j \leq 2d-2n} = ((\psi_i, \psi_j)(a))_{1 \leq i, j \leq 2d-2n}$$

is invertible. Then a subspace $H_1 \subset (L_W^2(\mathcal{I}))^2$ is an SSE of H_0 if and only if there exist two matrices $M_{d \times 2n}$ and $N_{d \times (2d-2n)}$ such that

$$\text{rank}(M, N) = d, \quad MJM^* = N\Lambda^T N^*, \quad (2.9)$$

and

$$H_1 = \{(y, \tilde{g}) \in H : My(a) - N \begin{pmatrix} (y, \psi_1)(+\infty) \\ \vdots \\ (y, \psi_{2d-2n})(+\infty) \end{pmatrix} = 0\}. \quad (2.10)$$

Remark 2.2. By [25, Theorem 4.2] one can rearrange ψ_1, \dots, ψ_d such that Λ is invertible, where ψ_1, \dots, ψ_d and Λ are specified in Lemma 2.8.

2.3 Resolvent convergence, spectral inclusion, and spectral exactness

In this subsection, we recall some basic concepts, including spectral inclusion, spectral exactness, and strong resolvent convergence for self-adjoint subspaces and list some useful results.

Definition 2.2 [30, Definition 4.1]. Let $\{T_k\}_{k=1}^\infty$ and T be self-adjoint subspaces in X^2 . $\{T_k\}_{k=1}^\infty$ is said to converge to T in the strong resolvent sense (briefly, SRC) if for some $\lambda \in \mathbf{C} \setminus \mathbf{R}$, $(\lambda I - T_k)^{-1}$ is strongly convergent to $(\lambda I - T)^{-1}$; that is, $\|(\lambda I - T_k)^{-1}f - (\lambda I - T)^{-1}f\| \rightarrow 0$ as $k \rightarrow \infty$ for any $f \in X$, denoted by $(\lambda I - T_k)^{-1} \xrightarrow{s} (\lambda I - T)^{-1}$.

Definition 2.3 [30, Definition 5.1]. Let $\{T_k\}_{k=1}^\infty$ and T be subspaces in X^2 .

- (1) The sequence $\{T_k\}_{k=1}^\infty$ is said to be spectrally inclusive for T if for any $\lambda \in \sigma(T)$, there exists a sequence $\{\lambda_k\}_{k=1}^\infty$, $\lambda_k \in \sigma(T_k)$, such that $\lim_{k \rightarrow \infty} \lambda_k = \lambda$.
- (2) The sequence $\{T_k\}_{k=1}^\infty$ is said to be spectrally exact for T if it is spectrally inclusive and every limit point of any sequence $\{\lambda_k\}_{k=1}^\infty$ with $\lambda_k \in \sigma(T_k)$ belongs to $\sigma(T)$.

The following result gives a sufficient condition for resolvent convergence of sequences of self-adjoint subspaces in the strong sense.

Lemma 2.9 [30, Theorem 4.2]. Let $\{T_k\}_{k=1}^\infty$ and T be self-adjoint subspaces in X^2 . Then $\{T_k\}$ is SRC to T if T has a core T_0 satisfying that $T_0 = \lim_{k \rightarrow \infty} T_k$; that is, for any $(x, f) \in T_0$, there exists $(x_k, f_k) \in T_k$ such that $(x, f) = \lim_{k \rightarrow \infty} (x_k, f_k)$.

A subspace T_0 is called a core of a closed subspace T if $\overline{T_0} = T$ (see Definition 3.3 in [29]).

The following result gives a sufficient condition for spectral inclusion and spectral exactness of a sequence of self-adjoint subspaces, which will take an important role in the study of regular approximations of spectrum.

Lemma 2.10 [30, Theorem 5.4]. Let $X_k, k \geq 1$, be proper closed subspaces in X , $P_k : X \rightarrow X_k$ orthogonal projections, and T and $\{T_k\}_{k=1}^\infty$ self-adjoint subspaces in X^2 and X_k^2 , respectively. Assume that $0 \notin \sigma(T) \neq \emptyset$ and $\sigma(T_k) \neq \emptyset$ for $k \geq 1$, and set $T'_k := T_k G(P_k)$. If $\{T'_k\}_{k=1}^\infty$ is SRC to T , then $\{T_k\}_{k=1}^\infty$ is spectrally inclusive for T . Further, if for any $\lambda \in \mathbf{C} \setminus \mathbf{R}$, $\|(\lambda I - T_k)^{-1} G(P_k) - (\lambda I - T)^{-1}\| \rightarrow 0$ as $k \rightarrow \infty$, denoted by $(\lambda I - T_k)^{-1} G(P_k) \xrightarrow{n} (\lambda I - T)^{-1}$, then $\{T_k\}_{k=1}^\infty$ is spectrally exact for T .

3 Constructing induced regular self-adjoint subspace extensions

Let $\mathcal{I}_r = \{t\}_{t=a}^{b_r}$, where $a < t_0 < b_r < +\infty$, $b_r \leq b_{r+1}$, $r \in \mathbf{Z}^+$, and $b_r \rightarrow +\infty$ as $r \rightarrow \infty$, where t_0 is specified by (\mathbf{A}_2) . For convenience, by H^r and H_0^r denote the corresponding maximal and minimal subspaces corresponding to system (1.1) or \mathcal{L} on \mathcal{I}_r , respectively.

Our main object in this section is to construct proper induced regular SSEs $H_{1,r}$ of \mathcal{L} on \mathcal{I}_r for any given SSE H_1 of H_0 . We shall use the spectra of $H_{1,r}$ to approximate the spectrum of the given SSE H_1 . The discussions are divided into the following three cases: \mathcal{L} is in l.c.c., l.p.c., and the intermediate cases at $t = +\infty$.

Case 1. The limit circle case

Let \mathcal{L} be in l.c.c. at $t = +\infty$. And let $\theta_1, \theta_2, \dots, \theta_{2n}$ be defined in Lemma 2.6. Set

$$\Theta(t, \lambda) = (\theta_1(t, \lambda), \theta_2(t, \lambda), \dots, \theta_{2n}(t, \lambda)). \quad (3.1)$$

Then by (2.2) and (2.4) we get that

$$\Theta^*(t, \lambda) J \Theta(t, \lambda) = J. \quad (3.2)$$

Suppose that H_1 is any fixed SSE of H_0 and characterized by (2.6), and matrices M, N satisfy (2.5). Let

$$JM^* = (\rho_1, \rho_2, \dots, \rho_{2n}), \quad N = (n_{ij})_{2n \times 2n}, \quad (3.3)$$

and $\varphi_i := \sum_{j=1}^{2n} \bar{n}_{ij} \theta_j$, $1 \leq i \leq 2n$. It is evident that $\varphi_i \in D(H)$, $1 \leq i \leq 2n$. By Lemma 2.4 there exist $\beta_i := (\omega_i, \tilde{\tau}_i) \in H$ ($1 \leq i \leq 2n$) such that

$$\omega_i(a) = \rho_i, \quad \omega_i(t) = \varphi_i(t), \quad t \geq t_0 + 1, \quad (3.4)$$

where t_0 is specified by (\mathbf{A}_2) . By noting that

$$My(a) = (JM^*)^* Jy(a) = \begin{pmatrix} \omega_1^*(a) \\ \vdots \\ \omega_{2n}^*(a) \end{pmatrix} Jy(a) = \begin{pmatrix} (y, \omega_1)(a) \\ \vdots \\ (y, \omega_{2n})(a) \end{pmatrix},$$

$$N \begin{pmatrix} (y, \theta_1)(+\infty) \\ \vdots \\ (y, \theta_{2n})(+\infty) \end{pmatrix} = \begin{pmatrix} (y, \sum_{j=1}^{2n} \bar{n}_{1j} \theta_j)(+\infty) \\ \vdots \\ (y, \sum_{j=1}^{2n} \bar{n}_{(2n)j} \theta_j)(+\infty) \end{pmatrix} = \begin{pmatrix} (y, \omega_1)(+\infty) \\ \vdots \\ (y, \omega_{2n})(+\infty) \end{pmatrix},$$

H_1 in (2.6) can be rewritten as the following form:

$$H_1 = \left\{ (y, \tilde{g}) \in H : \begin{pmatrix} (y, \omega_1)(a) \\ \vdots \\ (y, \omega_{2n})(a) \end{pmatrix} - \begin{pmatrix} (y, \omega_1)(+\infty) \\ \vdots \\ (y, \omega_{2n})(+\infty) \end{pmatrix} = 0 \right\}. \quad (3.5)$$

It can be easily verified that the set $\{\beta_i\}_{i=1}^{2n}$ is a GKN-set for $\{H_0, H_0^*\}$. For the definition of a GKN-set of Hermitian subspaces, the reader is referred to [29, Definition 4.1].

Next, we construct a proper induced regular SSE for \mathcal{L} on \mathcal{I}_r corresponding to the given SSE H_1 .

Let $b = b_r$, $P = M$, and $Q = N\Theta^*(b_r + 1)J$ in Lemma 2.5. Then (\mathbf{A}_2) for (1.1) on \mathcal{I}_r holds. Since Θ and J are invertible, one has that

$$\text{rank}(P, Q) = \text{rank}(M, N) = 2n.$$

By (3.2) we have

$$QJQ^* = N\Theta^*(b_r + 1)J\Theta(b_r + 1)N^* = NJN^*, \quad PJP^* = MJM^*.$$

Therefore,

$$PJP^* = QJQ^*.$$

In addition, because

$$\begin{pmatrix} (y, \theta_1)(t) \\ \vdots \\ (y, \theta_{2n})(t) \end{pmatrix} = \Theta^*(t)Jy(t),$$

the subspace

$$H_{1,r} = \left\{ (y, \tilde{g}) \in H^r : My(a) - N \begin{pmatrix} (y, \theta_1)(b_r + 1) \\ \vdots \\ (y, \theta_{2n})(b_r + 1) \end{pmatrix} = 0 \right\} \quad (3.6)$$

is an SSE of H_0^r by Lemma 2.5. With a similar argument to that used in the above discussion for (3.5), one can easily get that $H_{1,r}$ can be rewritten as

$$H_{1,r} = \left\{ (y, \tilde{g}) \in H^r : \begin{pmatrix} (y, \omega_1)(a) \\ \vdots \\ (y, \omega_{2n})(a) \end{pmatrix} - \begin{pmatrix} (y, \omega_1)(b_r + 1) \\ \vdots \\ (y, \omega_{2n})(b_r + 1) \end{pmatrix} = 0 \right\}. \quad (3.7)$$

We call $H_{1,r}$ an induced regular SSE of H_1 on \mathcal{I}_r . Further, it can be easily verified that $\{\beta_i|_{\mathcal{I}_r^+} := \{(\omega_i, \tilde{\tau}_i)(t)\}_{t=a}^{b_r+1}\}_{i=1}^{2n}$ is a GKN-set for $\{H_0^r, H_0^{r*}\}$.

Case 2. The limit point case

Let \mathcal{L} be in l.p.c. at $t = +\infty$. Suppose that H_1 is any fixed SSE of H_0 and characterized by (2.8), and the matrix $M_{n \times 2n}$ satisfies (2.7). Let

$$JM^* = (\alpha_1, \dots, \alpha_n).$$

By Lemma 2.4, there exist $\beta_i = (\omega_i, \tilde{\tau}_i) \in H$, $1 \leq i \leq n$, satisfying

$$\omega_i(a) = \alpha_i, \quad \omega_i(t) = 0, \quad t \geq t_0 + 1, \quad (3.8)$$

where t_0 is specified by (\mathbf{A}_2) . It follows that

$$My(a) = (JM^*)^* Jy(a) = \begin{pmatrix} \omega_1^*(a) \\ \vdots \\ \omega_n^*(a) \end{pmatrix} Jy(a) = \begin{pmatrix} (y, \omega_1)(a) \\ \vdots \\ (y, \omega_n)(a) \end{pmatrix},$$

which implies that H_1 in (2.8) can be written as

$$H_1 = \{(y, \tilde{g}) \in H : \begin{pmatrix} (y, \omega_1)(a) \\ \vdots \\ (y, \omega_n)(a) \end{pmatrix} = 0\}. \quad (3.9)$$

It is obvious that $\{\beta_i\}_{i=1}^n$ is a GKN-set for $\{H_0, H_0^*\}$.

Now, we construct a proper regular SSE $H_{1,r}$, which is induced by H_1 on \mathcal{I}_r . Set

$$P = \begin{pmatrix} M_{n \times 2n} \\ 0_{n \times 2n} \end{pmatrix}, \quad Q = - \begin{pmatrix} 0_{n \times 2n} \\ N_{n \times 2n} \end{pmatrix} \Theta^*(b_r + 1, \lambda) J,$$

where $N = (n_{ij})_{n \times 2n}$ with $\text{rank} N = n$ and $NJN^* = 0$, is any fixed matrix, Θ is defined by (3.1), and $\lambda \in \mathbf{R}$ is any fixed number. It can be easily verified that

$$\text{rank}(P, Q) = 2n, \quad PJP^* = QJQ^* = 0.$$

Further, it follows that

$$\begin{aligned} Py(a) - Qy(b_r + 1) &= \begin{pmatrix} M_{n \times 2n} \\ 0_{n \times 2n} \end{pmatrix} y(a) + \begin{pmatrix} 0_{n \times 2n} \\ N_{n \times 2n} \end{pmatrix} \Theta^*(b_r + 1) Jy(b_r + 1) \\ &= \begin{pmatrix} M_{n \times 2n} \\ 0_{n \times 2n} \end{pmatrix} y(a) + \begin{pmatrix} 0_{n \times 2n} \\ N_{n \times 2n} \end{pmatrix} \begin{pmatrix} (y, \theta_1)(b_r + 1) \\ \vdots \\ (y, \theta_{2n})(b_r + 1) \end{pmatrix} \\ &= \begin{pmatrix} My(a) \\ N \begin{pmatrix} (y, \theta_1)(b_r + 1) \\ \vdots \\ (y, \theta_{2n})(b_r + 1) \end{pmatrix} \end{pmatrix}. \end{aligned}$$

Therefore, by Lemma 2.5 one has that

$$H_{1,r} = \left\{ (y, \tilde{g}) \in H^r : My(a) = 0, \quad N \begin{pmatrix} (y, \theta_1)(b_r + 1) \\ \vdots \\ (y, \theta_{2n})(b_r + 1) \end{pmatrix} = 0 \right\} \quad (3.10)$$

is an SSE of H_0^r . Let

$$\varphi_i := \sum_{j=1}^{2n} \bar{n}_{ij} \theta_j, \quad 1 \leq i \leq n. \quad (3.11)$$

Similarly to the discussion for (3.5), $H_{1,r}$ can be rewritten as

$$H_{1,r} = \left\{ (y, \tilde{g}) \in H^r : \begin{pmatrix} (y, \omega_1)(a) \\ \vdots \\ (y, \omega_n)(a) \end{pmatrix} = 0, \quad \begin{pmatrix} (y, \varphi_1)(b_r + 1) \\ \vdots \\ (y, \varphi_n)(b_r + 1) \end{pmatrix} = 0 \right\}. \quad (3.12)$$

We call $H_{1,r}$ an induced regular SSE of H_1 on \mathcal{I}_r .

Case 3. The intermediate cases

Let \mathcal{L} be in the intermediate case at $t = +\infty$ with $n < d < 2n$. In the case, we always assume that

(A₄) There exists $\lambda_0 \in \mathbf{R}$ such that (1.1_{λ_0}) has d linear independent solutions in $\mathcal{L}_W^2(\mathcal{I})$.

Then we assert that (1.1_{λ_0}) has d linear independent solutions ψ_1, \dots, ψ_d in $\mathcal{L}_W^2(\mathcal{I})$ such that

$$\begin{aligned} \Lambda &:= ((\psi_i, \psi_j)(+\infty))_{1 \leq i, j \leq 2d-2n} \text{ is a diagonal and invertible matrix;} \\ ((\psi_i, \psi_j)(+\infty))_{1 \leq i, j \leq d} &= \begin{pmatrix} \Lambda & 0_{(2d-2n) \times (2n-d)} \\ 0_{(2n-d) \times (2d-2n)} & 0_{(2n-d) \times (2n-d)} \end{pmatrix}. \end{aligned} \quad (3.13)$$

In fact, let $\tilde{\psi}_1, \dots, \tilde{\psi}_d$ be any d linear independent solutions of (1.1_{λ_0}) in $\mathcal{L}_W^2(\mathcal{I})$. Let $\tilde{\Psi}_1 := (\tilde{\psi}_1, \dots, \tilde{\psi}_d)$. Then $\tilde{\Psi}_1^*(t)J\tilde{\Psi}_1(t) = \tilde{\Psi}_1^*(+\infty)J\tilde{\Psi}_1(+\infty)$ by (2.2), which is a skew-Hermitian matrix. In addition, $\text{rank}(\tilde{\Psi}_1^*(+\infty)J\tilde{\Psi}_1(+\infty)) = 2d - 2n$ by [25, Lemma 4.4]. Thus, there exists a unitary matrix U such that

$$U^*(\tilde{\Psi}_1^*(+\infty)J\tilde{\Psi}_1(+\infty))U = \begin{pmatrix} \tilde{\Lambda}_{(2d-2n) \times (2d-2n)} & 0_{(2d-2n) \times (2n-d)} \\ 0_{(2n-d) \times (2d-2n)} & 0_{(2n-d) \times (2n-d)} \end{pmatrix},$$

where $\tilde{\Lambda}_{(2d-2n) \times (2d-2n)}$ is a diagonal and invertible matrix. Let $\Psi_1 = (\psi_1, \dots, \psi_d) := \tilde{\Psi}_1 U$. Then

$$((\psi_i, \psi_j)(+\infty))_{1 \leq i, j \leq d} = (\Psi_1^*(+\infty)J\Psi_1(+\infty))^T = \begin{pmatrix} \tilde{\Lambda}_{(2d-2n) \times (2d-2n)} & 0_{(2d-2n) \times (2n-d)} \\ 0_{(2n-d) \times (2d-2n)} & 0_{(2n-d) \times (2n-d)} \end{pmatrix}$$

and so ψ_1, \dots, ψ_d are d linear independent solutions of (1.1_{λ_0}) in $\mathcal{L}_W^2(\mathcal{I})$ and satisfy (3.13). Thus, this assertion holds. In this case, we shall use these solutions ψ_1, \dots, ψ_d to characterize the self-adjoint subspace extensions H_1 of H_0 in Lemma 2.8.

Suppose that H_1 is any fixed SSE of H_0 and characterized by (2.10), and matrices M, N satisfy (2.9). Let

$$JM^* = (\gamma_1, \dots, \gamma_d), \quad N = (n_{ij})_{d \times (2d-2n)}, \quad (3.14)$$

and set $\varphi_i := \sum_{j=1}^{2d-2n} \bar{n}_{ij} \psi_j$, $1 \leq i \leq d$. Clearly, $\varphi_i \in D(H)$, $1 \leq i \leq d$. By Lemma 2.4 there exist $\beta_i := (\omega_i, \tilde{\tau}_i) \in H$ ($1 \leq i \leq d$) such that

$$\omega_i(a) = \gamma_i, \quad \omega_i(t) = \varphi_i(t), \quad t \geq t_0 + 1, \quad (3.15)$$

where t_0 is specified by (A₂). Note that for any $y \in D(H)$, it follows that

$$My(a) = (JM^*)^* Jy(a) = \begin{pmatrix} \omega_1^*(a) \\ \vdots \\ \omega_d^*(a) \end{pmatrix} Jy(a) = \begin{pmatrix} (y, \omega_1)(a) \\ \vdots \\ (y, \omega_d)(a) \end{pmatrix},$$

$$N \begin{pmatrix} (y, \psi_1)(+\infty) \\ \vdots \\ (y, \psi_{2d-2n})(+\infty) \end{pmatrix} = \begin{pmatrix} (y, \sum_{j=1}^{2d-2n} \bar{n}_{1j} \psi_j)(+\infty) \\ \vdots \\ (y, \sum_{j=1}^{2d-2n} \bar{n}_{dj} \psi_j)(+\infty) \end{pmatrix} = \begin{pmatrix} (y, \omega_1)(+\infty) \\ \vdots \\ (y, \omega_d)(+\infty) \end{pmatrix}.$$

Hence, H_1 in (2.10) can be rewritten as the following form:

$$H_1 = \left\{ (y, \tilde{g}) \in H : \begin{pmatrix} (y, \omega_1)(a) \\ \vdots \\ (y, \omega_d)(a) \end{pmatrix} - \begin{pmatrix} (y, \omega_1)(+\infty) \\ \vdots \\ (y, \omega_d)(+\infty) \end{pmatrix} = 0 \right\}. \quad (3.16)$$

It can be easily verified that the set $\{\beta_i\}_{i=1}^d$ is a GKN-set for $\{H_0, H_0^*\}$.

Next, we construct a proper induced regular SSE for \mathcal{L} on \mathcal{I}_r corresponding to the given SSE H_1 .

We still use the solutions ψ_1, \dots, ψ_d in $\mathcal{L}_W^2(\mathcal{I})$, which satisfy (3.13). In addition, we add solutions $\psi_{d+1}, \dots, \psi_{2n}$ such that $\{\psi_1, \dots, \psi_{2n}\}$ forms a basis of solutions of (1.1 $_{\lambda_0}$). Let $\Psi = (\psi_1, \dots, \psi_{2n})$. Then Ψ is obviously invertible. Set

$$P = \begin{pmatrix} M_{d \times 2n} \\ 0_{(2n-d) \times 2n} \end{pmatrix}, \quad Q = \begin{pmatrix} N_{d \times (2d-2n)} & 0_{d \times (2n-d)} & 0_{d \times (2n-d)} \\ 0_{(2n-d) \times (2d-2n)} & I_{2n-d} & 0_{(2n-d) \times (2n-d)} \end{pmatrix} \Psi^*(b_r + 1)J.$$

It is obvious that

$$\text{rank}(P, Q) = \text{rank}(M, N) + 2n - d = 2n.$$

By (2.2), (3.13), and $MJM^* = N\Lambda^T N^*$ we have

$$PJP^* = QJQ^*.$$

Further, it follows that

$$Py(a) - Qy(b_r + 1) = \begin{pmatrix} My(a) - N \begin{pmatrix} (y, \psi_1)(b_r + 1) \\ \vdots \\ (y, \psi_{2d-2n})(b_r + 1) \end{pmatrix} \\ - \begin{pmatrix} (y, \psi_{2d-2n+1})(b_r + 1) \\ \vdots \\ (y, \psi_d)(b_r + 1) \end{pmatrix} \end{pmatrix}.$$

Therefore, by Lemma 2.5 one has that

$$H_{1,r} = \left\{ (y, \tilde{g}) \in H^r : My(a) - N \begin{pmatrix} (y, \psi_1)(b_r + 1) \\ \vdots \\ (y, \psi_{2d-2n})(b_r + 1) \end{pmatrix} = 0, \begin{pmatrix} (y, \psi_{2d-2n+1})(b_r + 1) \\ \vdots \\ (y, \psi_d)(b_r + 1) \end{pmatrix} = 0 \right\}$$

is an SSE of H_0^r . Similarly to the discussion for (3.5), one can easily get that $H_{1,r}$ can be rewritten as

$$H_{1,r} = \left\{ (y, \tilde{g}) \in H^r : \begin{pmatrix} (y, \omega_1)(a) \\ \vdots \\ (y, \omega_d)(a) \end{pmatrix} = \begin{pmatrix} (y, \omega_1)(b_r + 1) \\ \vdots \\ (y, \omega_d)(b_r + 1) \end{pmatrix}, \begin{pmatrix} (y, \psi_{2d-2n+1})(b_r + 1) \\ \vdots \\ (y, \psi_d)(b_r + 1) \end{pmatrix} = 0 \right\}, \quad (3.17)$$

where $\omega_1, \dots, \omega_d$ are defined by (3.15). We call $H_{1,r}$ an induced regular SSE of H_1 on \mathcal{I}_r .

4 Extension of the induced regular self-adjoint subspace extensions to the whole space

In this section, we first extend a subspace in the product space of the fundamental spaces on a proper subinterval to a subspace in that on the original interval, and study spectral properties of the extended subspaces. As a consequence, the extension from the induced regular SSE constructed in Section 3 to a subspace in $(L_W^2(\mathcal{I}))^2$ is given, and the spectral properties of the extended subspaces are obtained.

Let $\mathcal{K} := \{t\}_{t=a}^b$ be an integer interval, where a is a finite integer or $a = -\infty$ and b is a finite integer or $b = +\infty$. \mathcal{K}^+ , $\mathcal{L}_W^2(\mathcal{K})$, and $L_W^2(\mathcal{K})$ can be well defined as in Section 2.2 with \mathcal{I} replaced by \mathcal{K} . For convenience, by $\langle \cdot, \cdot \rangle_{\mathcal{K}}$ and $\|\cdot\|_{\mathcal{K}}$ denote the inner product and norm of $(L_W^2(\mathcal{K}))^2$, respectively.

For any integer interval $\mathcal{J} \subsetneq \mathcal{K}$, denote

$$\hat{L}_W^2(\mathcal{J}) := \{\tilde{y} \in L_W^2(\mathcal{K}) : W(t)R(y)(t) = 0, \ t \in \mathcal{K} \setminus \mathcal{J}\}. \quad (4.1)$$

For any subspace T in $(L_W^2(\mathcal{J}))^2$, denote

$$\hat{T} := \{(\tilde{y}, \tilde{g}) \in (\hat{L}_W^2(\mathcal{J}))^2 : \text{there exists } (y, g) \in T \text{ such that} \\ \|y\|_{\mathcal{J}} = \|\tilde{y}\|_{\mathcal{K}}, \ \|g\|_{\mathcal{J}} = \|\tilde{g}\|_{\mathcal{K}}\}. \quad (4.2)$$

The following result can be easily verified, and so its details are omitted.

Proposition 4.1. $\hat{L}_W^2(\mathcal{J})$ is a closed subspace in $L_W^2(\mathcal{K})$ and so \hat{T} is a subspace in $(L_W^2(\mathcal{K}))^2$. Moreover, $\langle x, y \rangle_{\mathcal{K}} = \langle x|_{\mathcal{J}^+}, y|_{\mathcal{J}^+} \rangle_{\mathcal{J}}$ for any $\tilde{x}, \tilde{y} \in \hat{L}_W^2(\mathcal{J})$.

Proposition 4.2. Let T be a subspace in $(L_W^2(\mathcal{J}))^2$ and \hat{T} be defined by (4.2). Then

- (i) T is a closed subspace in $(L_W^2(\mathcal{J}))^2$ if and only if \hat{T} is a closed subspace in $(\hat{L}_W^2(\mathcal{J}))^2$;
- (ii) T is a Hermitian subspace in $(L_W^2(\mathcal{J}))^2$ if and only if \hat{T} is a Hermitian subspace in $(\hat{L}_W^2(\mathcal{J}))^2$;
- (iii) T is a self-adjoint subspace in $(L_W^2(\mathcal{J}))^2$ if and only if \hat{T} is a self-adjoint subspace in $(\hat{L}_W^2(\mathcal{J}))^2$.

Proof. (i) Assertion (i) can be directly derived from (4.2) and Proposition 4.1.

(ii) We first show the necessity. Suppose that T is a Hermitian subspace in $(L_W^2(\mathcal{J}))^2$. For any $(\tilde{y}_i, \tilde{g}_i) \in \hat{T}$, $i = 1, 2$, by (4.2) there exist $(y_i, g_i) \in T$, $i = 1, 2$, such that $\|y_i\|_{\mathcal{J}} = \|\tilde{y}_i\|_{\mathcal{K}}$ and $\|g_i\|_{\mathcal{J}} = \|\tilde{g}_i\|_{\mathcal{K}}$, $i = 1, 2$, which are equivalent to

$$W(t)R(y_i)(t) = W(t)R(\tilde{y}_i)(t), \ W(t)R(g_i)(t) = W(t)R(\tilde{g}_i)(t), \ t \in \mathcal{J}, \ i = 1, 2. \quad (4.3)$$

Since T is Hermitian,

$$\langle g_1, y_2 \rangle_{\mathcal{J}} = \langle y_1, g_2 \rangle_{\mathcal{J}}. \quad (4.4)$$

This, together with (4.1)-(4.3), yields that

$$\langle \hat{g}_1, \hat{y}_2 \rangle_{\mathcal{K}} = \langle \hat{y}_1, \hat{g}_2 \rangle_{\mathcal{K}}. \quad (4.5)$$

This implies that \hat{T} is a Hermitian subspace in $(\hat{L}_W^2(\mathcal{J}))^2$.

Next we consider the sufficiency. Suppose that \hat{T} is a Hermitian subspace in $(\hat{L}_W^2(\mathcal{J}))^2$. For any $(\tilde{y}_i, \tilde{g}_i) \in T$, $i = 1, 2$, take $\tilde{\hat{y}}_i, \tilde{\hat{g}}_i \in \hat{L}_W^2(\mathcal{J})$, $i = 1, 2$, such that (4.3) holds. Then $(\tilde{\hat{y}}_i, \tilde{\hat{g}}_i) \in \hat{T}$, $i = 1, 2$. Since \hat{T} is Hermitian, one has that (4.5) holds. This, together with (4.1) and (4.3), yields that (4.4) holds. This implies that T is a Hermitian subspace in $(L_W^2(\mathcal{J}))^2$.

(iii) We first show the necessity. Suppose that T is a self-adjoint subspace in $(L_W^2(\mathcal{J}))^2$. By (ii) we get that \hat{T} is a Hermitian subspace in $(\hat{L}_W^2(\mathcal{J}))^2$. So, it is only needed to show that $\hat{T}^* \subset \hat{T}$. By the definition of adjoint subspace, for any given $(\tilde{\hat{y}}_1, \tilde{\hat{g}}_1) \in \hat{T}^*$, (4.5) holds for all $(\tilde{\hat{y}}_2, \tilde{\hat{g}}_2) \in \hat{T}$. Set $y_1 := \hat{y}_1|_{\mathcal{J}^+}$ and $g_1 := \hat{g}_1|_{\mathcal{J}^+}$. Then $\tilde{y}_1, \tilde{g}_1 \in L_W^2(\mathcal{J})$ and (4.3) holds for $i = 1$. In addition, for any $(\tilde{y}_2, \tilde{g}_2) \in T$, there exists $(\tilde{\hat{y}}_2, \tilde{\hat{g}}_2) \in \hat{T}$ such that (4.3) holds for $i = 2$. Hence, we get that $\langle \hat{g}_1, \hat{y}_2 \rangle_{\mathcal{K}} = \langle g_1, y_2 \rangle_{\mathcal{J}}$ and $\langle \hat{y}_1, \hat{g}_2 \rangle_{\mathcal{K}} = \langle y_1, g_2 \rangle_{\mathcal{J}}$. It follows from (4.5) that

$$\langle g_1, y_2 \rangle_{\mathcal{J}} = \langle y_1, g_2 \rangle_{\mathcal{J}}, \quad \forall (\tilde{y}_2, \tilde{g}_2) \in T. \quad (4.6)$$

Because T is a self-adjoint subspace in $(L_W^2(\mathcal{J}))^2$, we get that $(\tilde{y}_1, \tilde{g}_1) \in T$. Therefore, $(\tilde{\hat{y}}_1, \tilde{\hat{g}}_1) \in \hat{T}$ by (4.2). This implies that $\hat{T}^* \subset \hat{T}$. Thus, \hat{T} is a self-adjoint subspace in $(\hat{L}_W^2(\mathcal{J}))^2$.

Next we consider the sufficiency. Suppose that \hat{T} is a self-adjoint subspace in $(\hat{L}_W^2(\mathcal{J}))^2$. Similarly, by (ii) we only need to show that $T^* \subset T$. By the definition of adjoint subspace, for any given $(\tilde{y}_1, \tilde{g}_1) \in T^*$, (4.6) holds. Take $\tilde{\hat{y}}_1, \tilde{\hat{g}}_1 \in \hat{L}_W^2(\mathcal{J})$ such that (4.3) holds for $i = 1$. In addition, for any $(\tilde{\hat{y}}_2, \tilde{\hat{g}}_2) \in \hat{T}$, by (4.2) there exists $(\tilde{y}_2, \tilde{g}_2) \in T$ such that (4.3) holds for $i = 2$. It follows from (4.3) and (4.6) that (4.5) holds for all $(\tilde{\hat{y}}_2, \tilde{\hat{g}}_2) \in \hat{T}$. This implies that $(\tilde{\hat{y}}_1, \tilde{\hat{g}}_1) \in \hat{T}^* = \hat{T}$. This, together with (4.2) and (4.3) with $i = 1$, yields that $(\tilde{y}_1, \tilde{g}_1) \in T$. Therefore, $T^* \subset T$ and so T is a self-adjoint subspace in $(L_W^2(\mathcal{J}))^2$. The whole proof is complete.

Proposition 4.3. *Let T be a closed subspace in $(L_W^2(\mathcal{J}))^2$ and \hat{T} be defined by (4.2). Then $\sigma(\hat{T}) = \sigma(T)$.*

Proof. By Lemma 2.2, it suffices to show that

$$N(\lambda I - \hat{T}) \neq \{0\} \Leftrightarrow N(\lambda I - T) \neq \{0\}, \quad (4.7)$$

$$R(\lambda I - \hat{T}) = \hat{L}_W^2(\mathcal{J}) \Leftrightarrow R(\lambda I - T) = L_W^2(\mathcal{J}). \quad (4.8)$$

We first show that (4.7) holds. Suppose that $N(\lambda I - T) \neq \{0\}$. Then there exists $\tilde{x} \in D(T)$ with $\|\tilde{x}\|_{\mathcal{J}} > 0$ such that $(\tilde{x}, 0) \in \lambda I - T$, i.e., $(\tilde{x}, \lambda \tilde{x}) \in T$. Take $\tilde{\hat{x}} \in \hat{L}_W^2(\mathcal{J})$ satisfying

$\|\hat{x}\|_{\mathcal{K}} = \|x\|_{\mathcal{J}}$. Then, by (4.2), $(\tilde{x}, \lambda\tilde{x}) \in \hat{T}$, i.e., $(\tilde{x}, 0) \in \lambda I - \hat{T}$. So $N(\lambda I - \hat{T}) \neq \{0\}$ by $\|\hat{x}\|_{\mathcal{K}} > 0$. On the other hand, suppose that $N(\lambda I - \hat{T}) \neq \{0\}$. Then, there exists $\tilde{y} \in D(\hat{T})$ with $\|\tilde{y}\|_{\mathcal{K}} > 0$ such that $(\tilde{y}, 0) \in \lambda I - \hat{T}$, which implies that $(\tilde{y}, \lambda\tilde{y}) \in \hat{T}$. By (4.2), there exists $(\tilde{y}, \lambda\tilde{y}) \in T$ such that $\|y\|_{\mathcal{J}} = \|\tilde{y}\|_{\mathcal{K}}$. Thus, we have that $(\tilde{y}, 0) \in \lambda I - T$ and $\|y\|_{\mathcal{J}} > 0$. Hence, $N(\lambda I - T) \neq \{0\}$. Therefore, (4.7) holds.

Next, we show that (4.8) holds. Suppose that $R(\lambda I - T) = L_W^2(\mathcal{J})$. For any $\tilde{g} \in \hat{L}_W^2(\mathcal{J})$, set $g := \hat{g}|_{\mathcal{J}^+}$. Then $\tilde{g} \in L_W^2(\mathcal{J})$. There exists $\tilde{y} \in D(T)$ such that $(\tilde{y}, \tilde{g}) \in \lambda I - T$, i.e., $(\tilde{y}, \lambda\tilde{y} - \tilde{g}) \in T$. Take $\tilde{y} \in \hat{L}_W^2(\mathcal{J})$ satisfying $\|\tilde{y}\|_{\mathcal{K}} = \|\tilde{y}\|_{\mathcal{J}}$. Then $(\tilde{y}, \lambda\tilde{y} - \tilde{g}) \in \hat{T}$, i.e., $(\tilde{y}, \tilde{g}) \in \lambda I - \hat{T}$. This yields that $R(\lambda I - \hat{T}) = \hat{L}_W^2(\mathcal{J})$. On the other hand, suppose that $R(\lambda I - \hat{T}) = \hat{L}_W^2(\mathcal{J})$. For any $\tilde{f} \in L_W^2(\mathcal{J})$, take $\tilde{f} \in \hat{L}_W^2(\mathcal{J})$ satisfying $\|\tilde{f}\|_{\mathcal{K}} = \|\tilde{f}\|_{\mathcal{J}}$. Then, there exists $\tilde{x} \in D(\hat{T})$ such that $(\tilde{x}, \tilde{f}) \in \lambda I - \hat{T}$, i.e., $(\tilde{x}, \lambda\tilde{x} - \tilde{f}) \in \hat{T}$. Then, by (4.2), there exists $(\tilde{x}, \tilde{h}) \in T$ satisfying $\|x\|_{\mathcal{J}} = \|\tilde{x}\|_{\mathcal{K}}$ and $\|h\|_{\mathcal{J}} = \|\lambda\tilde{x} - \tilde{f}\|_{\mathcal{K}} = \|\lambda x - f\|_{\mathcal{J}}$. Consequently, $\tilde{h} = \lambda\tilde{x} - \tilde{f}$ and $(\tilde{x}, \tilde{f}) \in \lambda I - T$. Hence, $R(\lambda I - T) = L_W^2(\mathcal{J})$. Therefore, (4.8) holds. This completes the proof.

Remark 4.1. (4.1) and (4.2) are called the zero extensions in the discrete case. This problem can be easily solved in the continuous case but hard in the discrete case. For the zero extensions and their properties in analogy with those in Propositions 4.1-4.3 in the continuous case, please see [3, 7].

Note that H_1 and $H_{1,r}$ are self-adjoint subspaces in $(L_W^2(\mathcal{I}))^2$ and $(L_W^2(\mathcal{I}_r))^2$, respectively. It is difficult to study the convergence of $H_{1,r}$ to H_1 in some sense since $L_W^2(\mathcal{I})$ and $L_W^2(\mathcal{I}_r)$ are different spaces. In order to overcome this problem, we respectively extend $L_W^2(\mathcal{I}_r)$ and $H_{1,r}$ to be $\hat{L}_W^2(\mathcal{I}_r)$ and $\hat{H}_{1,r}$ by (4.1) and (4.2). Let P_r be the orthogonal projection from $L_W^2(\mathcal{I})$ to $\hat{L}_W^2(\mathcal{I}_r)$. Define

$$H'_{1,r} := \hat{H}_{1,r}G(P_r). \quad (4.9)$$

The following result gives the relationship between the spectra of $H'_{1,r}$, $\hat{H}_{1,r}$, and H_1 , which is a direct consequence of Propositions 4.2, 4.3, and Lemma 2.3.

Lemma 4.1. *Let H_1 be an SSE of H_0 , and $H_{1,r}$ the induced regular SSE of H_1 on \mathcal{I}_r . Then $\hat{H}_{1,r}$ and $H'_{1,r}$ are self-adjoint subspaces in $(\hat{L}_W^2(\mathcal{I}_r))^2$ and $(L_W^2(\mathcal{I}))^2$, respectively, $D(H'_{1,r}) = D(\hat{H}_{1,r}) \oplus (\hat{L}_W^2(\mathcal{I}_r))^\perp$, $\sigma(\hat{H}_{1,r}) = \sigma(H_{1,r})$, and $\sigma(H'_{1,r}) = \sigma(\hat{H}_{1,r}) \cup \{0\} = \sigma(H_{1,r}) \cup \{0\}$.*

5 Spectral approximation in the limit circle case

In this section, we shall study the regular approximation of spectra of (1.1) in the case that \mathcal{L} is in l.c.c. at $t = +\infty$. In this case, we shall show that $\{H_{1,r}\}_{r=1}^\infty$ is not only spectrally inclusive but also spectrally exact for any given H_1 . In addition, we obtain explicit

approximation relations and give their error estimates. We always assume that $(\mathbf{A}_1) - (\mathbf{A}_3)$ hold in this section.

For convenience, for any $\tilde{y} \in L_W^2(\mathcal{I})$ and for any $y \in \tilde{y}$, denote

$$\tilde{y}^r := P_r \tilde{y}, \quad y_r := y|_{\mathcal{I}_r^+}. \quad (5.1)$$

Then, $\tilde{y}^r \in \hat{L}_W^2(\mathcal{I}_r)$, $\tilde{y}_r \in L_W^2(\mathcal{I}_r)$, and

$$W(t)R(\tilde{y}^r)(t) = W(t)R(\tilde{y}_r)(t) = W(t)R(y)(t), \quad t \in \mathcal{I}_r. \quad (5.2)$$

Theorem 5.1. *Assume that \mathcal{L} is in l.c.c. at $t = +\infty$. Let H_1 be any fixed SSE of H_0 , and $H_{1,r}$ the induced regular SSE of H_1 on \mathcal{I}_r , where H_1 and $H_{1,r}$ are determined by (3.5) and (3.7), respectively. And let $H'_{1,r}$ be defined by (4.9). Then*

- (i) $\{H'_{1,r}\}$ is SRC to H_1 ;
- (ii) $\{H_{1,r}\}$ is spectrally inclusive for H_1 if $0 \notin \sigma(H_1)$.

Proof. The proof of assertion (i) is divided into three steps:

Step 1. Construct a core of H_1 .

Let

$$C(H_1) = H_{00} \dot{+} L\{\beta_1, \dots, \beta_{2n}\}, \quad (5.3)$$

where $\beta_i = (\omega_i, \tilde{\tau}_i) \in H$, $1 \leq i \leq 2n$, are given by (3.4). By the discussion for Case 1 in Section 3, $\{\beta_1, \dots, \beta_{2n}\}$ is a GKN-set for $\{H_0, H_0^*\}$. By [29, Theorem 4.2] one gets that

$$H_1 = H_0 \dot{+} L\{\beta_1, \dots, \beta_{2n}\},$$

which, together with the fact that $\overline{H}_{00} = H_0$, implies that $C(H_1)$ is a core of H_1 .

Step 2. For any $(y, \tilde{g}) \in C(H_1)$, there exists $r_0 \in \mathbf{Z}^+$ such that $(\tilde{y}, \tilde{g}^r) \in H'_{1,r}$ for all $r \geq r_0$.

In order to show that this assertion holds, it suffices to show that for any $(y, \tilde{g}) \in C(H_1)$, there exists $r_0 \in \mathbf{Z}^+$ such that $(y_r, \tilde{g}_r) \in H_{1,r}$ for all $r \geq r_0$. In fact, for each $(y, \tilde{g}) \in C(H_1)$, if $(y_r, \tilde{g}_r) \in H_{1,r}$, then $(\tilde{y}^r, \tilde{g}^r) \in \hat{H}_{1,r}$. In addition, since $(\tilde{y}, \tilde{y}^r) \in G(P_r)$, we have that $(\tilde{y}, \tilde{g}^r) \in H'_{1,r}$ by the definition of $H'_{1,r}$.

Note that (\mathbf{A}_2) for (1.1) on \mathcal{I}_r holds since $b_r > t_0$. For any given $(y, \tilde{g}) \in H_{00}$, by the definition of H_{00} , there exists $r_0 \in \mathbf{Z}^+$ such that $(y_r, \tilde{g}_r) \in H_{1,r}$ for all $r \geq r_0$. So it is only needed to show that for any $(y, \tilde{g}) \in L\{\beta_1, \dots, \beta_{2n}\}$, $(y_r, \tilde{g}_r) \in H_{1,r}$. For any given $(y, \tilde{g}) \in L\{\beta_1, \dots, \beta_{2n}\}$, there exist $d_1, \dots, d_{2n} \in \mathbf{C}$ such that

$$y = \sum_{i=1}^{2n} d_i \omega_i, \quad \tilde{g} = \sum_{i=1}^{2n} d_i \tilde{\tau}_i.$$

Since $(y, \tilde{g}) \in H_1$, by (3.5) we get that

$$(y, \omega_i)(a) - (y, \omega_i)(+\infty) = 0, \quad 1 \leq i \leq 2n. \quad (5.4)$$

In addition, since ω_i , $1 \leq i \leq 2n$, are solutions of (1.1 $_\lambda$) with $\lambda \in \mathbf{R}$ on $[t_0 + 1, +\infty)$ by (3.4), it follows from (2.2) that

$$(y, \omega_i)(t) = (y, \omega_i)(+\infty) \text{ for } t \geq t_0 + 1, \quad 1 \leq i \leq 2n. \quad (5.5)$$

Noting that $b_r > t_0$, by (5.4)-(5.5) one has that

$$(y, \omega_i)(a) - (y, \omega_i)(b_r + 1) = 0, \quad 1 \leq i \leq 2n,$$

which yields that $(y_r, \tilde{g}_r) \in H_{1,r}$ by (3.7). Hence, the assertion in this step holds.

Step 3. $\{H'_{1,r}\}$ and H_1 satisfy the conditions in Lemma 2.9.

It follows from Lemma 4.1 that $H'_{1,r}, r \geq 1$, are self-adjoint subspaces in $(L^2_W(\mathcal{I}))^2$. By the assertion in Step 2, we get that for any $(y, \tilde{g}) \in C(H_1)$, there exists a $r_0 \in \mathbf{Z}^+$ such that $(\tilde{y}, \tilde{g}^r) \in H'_{1,r}$ for $r \geq r_0$. Since $\|y\| = \|\tilde{y}\|$ and $\tilde{g} = \lim_{r \rightarrow \infty} \tilde{g}^r$, all the conditions in Lemma 2.9 are satisfied. Therefore, $\{H'_{1,r}\}$ is SRC to H_1 by Lemma 2.9.

Assertion (ii) can be directly derived from assertion (i) and Lemmas 2.10 and 4.1. This completes the proof.

Next, in order to show that $\{H_{1,r}\}$ is spectrally exact for H_1 , we shall give the explicit representations of the resolvents of H_1 and $H_{1,r}$ in terms of the Green functions, respectively, which will play an important role in the discussion of norm resolvent convergence (for the concept of norm resolvent convergence for self-adjoint subspaces, please see [30, Definition 4.1]), spectral exactness, and some other topics.

Proposition 5.1. *Assume that \mathcal{L} is in l.c.c. at $t = +\infty$. Let H_1 be any SSE of H_0 . For any $z \in \rho(H_1)$, let $\Phi(t, z) = (\phi_1, \dots, \phi_{2n})(t, z)$ be a standard fundamental solution matrix of (1.1_z) with $\Phi(a, z) = I_{2n}$. Then, for any $\tilde{g} \in L^2_W(\mathcal{I})$,*

$$y(t) = (zI - H_1)^{-1}(\tilde{g})(t) = \sum_{s=a}^{+\infty} G(t, s, z)W(s)R(g)(s), \quad t \in \mathcal{I}, \quad (5.6)$$

where

$$G(t, s, z) = \begin{cases} \Phi(t, z)M_0R(\Phi)^*(s, \bar{z}), & a \leq s < t < +\infty, \\ \Phi(t, z)N_0R(\Phi)^*(s, \bar{z}), & a \leq t \leq s < +\infty, \end{cases} \quad (5.7)$$

is called the Green function of the resolvent $(zI - H_1)^{-1}$, while M_0 and N_0 are determined by (5.9), (5.11), and (5.12).

Proof. For any fixed $z \in \rho(H_1)$ and for any given $\tilde{g} \in L^2_W(\mathcal{I})$, from $y = (zI - H_1)^{-1}\tilde{g}$, one has that $(y, \tilde{g}) \in zI - H_1$, and thus $(y, z\tilde{y} - \tilde{g}) \in H_1$, which implies that

$$\mathcal{L}(y)(t) = W(t)R(zy - g)(t), \quad t \in \mathcal{I};$$

that is,

$$J\Delta y(t) - P(t)R(y)(t) = zW(t)R(y)(t) - W(t)R(g)(t), \quad t \in \mathcal{I}.$$

By the variation of constants formula, every solution y can be given by

$$y(t) = \Phi(t, z)y(a) + \Phi(t, z)J \sum_{s=a}^{t-1} R(\Phi)^*(s, \bar{z})W(s)R(g)(s), \quad t \in \mathcal{I}, \quad (5.8)$$

where we promise that

$$\sum_{s=a}^{a-1} R(\Phi)^*(s, \bar{z})W(s)R(g)(s) = 0.$$

Denote

$$\Omega(t) := (\omega_1, \dots, \omega_{2n})(t), \quad (5.9)$$

where $\omega_i, 1 \leq i \leq 2n$, are defined by (3.4). In view of $y \in D(H_1)$, it follows from (3.5) that

$$\Omega^*(a)Jy(a) = \lim_{t \rightarrow +\infty} \Omega^*(t)Jy(t). \quad (5.10)$$

Inserting (5.8) into (5.10), we get that

$$\begin{aligned} & \Omega^*(a)Jy(a) \\ &= \lim_{t \rightarrow +\infty} \left\{ \left[\Omega^*(t)J\Phi(t, z) \right] y(a) + \left[\Omega^*(t)J\Phi(t, z) \right] J \sum_{s=a}^{t-1} R(\Phi)^*(s, \bar{z})W(s)R(g)(s) \right\}. \end{aligned}$$

Since \mathcal{L} is in l.c.c. at $t = +\infty$, we get that $\phi_1, \dots, \phi_{2n} \in \mathcal{L}_W^2(\mathcal{I})$, which, together with (2.1) and $g \in \mathcal{L}_W^2(\mathcal{I})$, yields that

$$\lim_{t \rightarrow +\infty} \sum_{s=a}^{t-1} R(\Phi)^*(s, \bar{z})W(s)R(g)(s) = \sum_{s=a}^{+\infty} R(\Phi)^*(s, \bar{z})W(s)R(g)(s),$$

and

$$K := \lim_{t \rightarrow +\infty} \Omega^*(t)J\Phi(t, z) \quad (5.11)$$

exist and are finite. So, we have

$$(\Omega^*(a)J - K)y(a) = KJ \sum_{s=a}^{+\infty} R(\Phi)^*(s, \bar{z})W(s)R(g)(s).$$

By the fact that $z \in \rho(H_1)$, it can be easily verified that $\Omega^*(a)J - K$ is invertible. Thus,

$$y(a) = (\Omega^*(a)J - K)^{-1} KJ \sum_{s=a}^{+\infty} R(\Phi)^*(s, \bar{z})W(s)R(g)(s).$$

Inserting it into (5.8), we get that

$$\begin{aligned} y(t) &= \Phi(t, z)(\Omega^*(a)J - K)^{-1} KJ \sum_{s=a}^{+\infty} R(\Phi)^*(s, \bar{z})W(s)R(g)(s) \\ &\quad + \Phi(t, z)J \sum_{s=a}^{t-1} R(\Phi)^*(s, \bar{z})W(s)R(g)(s) \\ &= \Phi(t, z)M_0 \sum_{s=a}^{t-1} R(\Phi)^*(s, \bar{z})W(s)R(g)(s) + \Phi(t, z)N_0 \sum_{s=t}^{+\infty} R(\Phi)^*(s, \bar{z})W(s)R(g)(s), \end{aligned}$$

where

$$M_0 = N_0 + J, \quad N_0 = (\Omega^*(a)J - K)^{-1} KJ. \quad (5.12)$$

Therefore, we can write

$$y(t) = \sum_{s=a}^{+\infty} G(t, s, z)W(s)R(g)(s),$$

where $G(t, s, z)$ is specified in (5.7). This completes the proof.

Next, consider the explicit representation of resolvent $(zI - H_{1,r})^{-1}$. With a similar argument in the proof of Proposition 5.1, one can easily show the following result:

Proposition 5.2. *For any $z \in \rho(H_{1,r})$ and for any $\tilde{g} \in L_W^2(\mathcal{I}_r)$,*

$$y(t) = (zI - H_{1,r})^{-1}(\tilde{g})(t) = \sum_{s=a}^{b_r} G_r(t, s, z)W(s)R(g)(s), \quad t \in \mathcal{I}_r, \quad (5.13)$$

where

$$G_r(t, s, z) = \begin{cases} \Phi(t, z)M_r R(\Phi)^*(s, \bar{z}), & a \leq s < t \leq b_r, \\ \Phi(t, z)N_r R(\Phi)^*(s, \bar{z}), & a \leq t \leq s \leq b_r, \end{cases} \quad (5.14)$$

is called the Green function of the resolvent $(zI - H_{1,r})^{-1}$, while M_r and N_r are determined by

$$M_r = N_r + J, \quad N_r = (\Omega^*(a)J - K_r)^{-1}K_r J, \quad K_r = \Omega^*(b_r + 1)J\Phi(b_r + 1, z), \quad (5.15)$$

where $\Omega(t)$ is specified by (5.9), and $\Phi(t, z)$ is specified in Proposition 5.1.

Let $A = (a_{ij}) \in \mathbf{C}^{k \times l}$ and $\xi = (\xi_1, \dots, \xi_l)^T \in \mathbf{C}^l$. Define their norms as

$$\|A\|_1 := \left(\sum_{i=1}^k \sum_{j=1}^l |a_{ij}|^2 \right)^{1/2}, \quad \|\xi\|_1 = \left(\sum_{j=1}^l |\xi_j|^2 \right)^{1/2}.$$

Then

$$\|A\xi\|_1 \leq \|A\|_1 \|\xi\|_1, \quad \|AB\|_1 \leq \|A\|_1 \|B\|_1, \quad \forall B \in \mathbf{C}^{l \times n}.$$

It follows from (5.11) and (5.15) that $K_r \rightarrow K$ as $r \rightarrow +\infty$. So one can get the following result by Propositions 5.1 and 5.2.

Proposition 5.3. *$M_r \rightarrow M_0$ and $N_r \rightarrow N_0$ as $r \rightarrow +\infty$.*

Now, we can give the following result about spectral exactness.

Theorem 5.2. *Assume that \mathcal{L} is in l.c.c. at $t = +\infty$. Let H_1 be any fixed SSE of H_0 , and $H_{1,r}$ the induced regular SSE of H_1 on \mathcal{I}_r , where H_1 and $H_{1,r}$ are determined by (3.5) and (3.7), respectively. And let $\hat{H}_{1,r}$ be defined by (4.2). Then*

- (i) *for any $z \in \rho(H_1) \cap \rho(\hat{H}_{1,r})$, $\{(zI - \hat{H}_{1,r})^{-1}G(P_r)\} \xrightarrow{n} (zI - H_1)^{-1}$;*
- (ii) *$\{H_{1,r}\}$ is spectrally exact for H_1 if $0 \notin \sigma(H_1)$.*

Proof. We first show that assertion (i) holds. Let $z \in \rho(H_1) \cap \rho(\hat{H}_{1,r})$. Note that $(zI - \hat{H}_{1,r})^{-1}$ is an operator. So we write $\{(zI - \hat{H}_{1,r})^{-1}G(P_r)\}$ as $\{(zI - \hat{H}_{1,r})^{-1}P_r\}$ for short. It follows from (5.1)-(5.2) that for any given $\tilde{g} \in L_W^2(\mathcal{I})$,

$$\begin{aligned} W(t)R((zI - \hat{H}_{1,r})^{-1}P_r\tilde{g})(t) &= W(t)R((zI - \hat{H}_{1,r})^{-1}\tilde{g}^r)(t) \\ &= W(t)R((zI - H_{1,r})^{-1}\tilde{g}^r)(t), \quad t \in \mathcal{I}_r. \end{aligned}$$

It yields that

$$\delta_r(\tilde{g}) := \|(zI - H_1)^{-1}\tilde{g} - (zI - \hat{H}_{1,r})^{-1}P_r\tilde{g}\|^2 = \delta_{r1}(\tilde{g}) + \delta_{r2}(\tilde{g}), \quad (5.16)$$

where

$$\begin{aligned} \delta_{r1}(\tilde{g}) &:= \sum_{t=a}^{b_r} R((zI - H_1)^{-1}\tilde{g} - (zI - H_{1,r})^{-1}\tilde{g}_r)^*(t)W(t)R((zI - H_1)^{-1}\tilde{g} - (zI - H_{1,r})^{-1}\tilde{g}_r)(t), \\ \delta_{r2}(\tilde{g}) &:= \sum_{t=b_r+1}^{+\infty} R((zI - H_1)^{-1}\tilde{g})^*(t)W(t)R((zI - H_1)^{-1}\tilde{g})(t). \end{aligned}$$

Now, we first consider $\delta_{r1}(\tilde{g})$. By Propositions 5.1 and 5.2 we get that

$$\begin{aligned} & (zI - H_1)^{-1}\tilde{g}(t) - (zI - H_{1,r})^{-1}\tilde{g}_r(t) \\ &= \sum_{s=a}^{+\infty} G(t, s, z)W(s)R(g)(s) - \sum_{s=a}^{b_r} G_r(t, s, z)W(s)R(g_r)(s) \\ &= T_{r1}(\tilde{g})(t) + T_{r2}(\tilde{g})(t) + T_{r3}(\tilde{g})(t), \quad t \in \mathcal{I}_r, \end{aligned}$$

where

$$\begin{aligned} T_{r1}(\tilde{g})(t) &:= \Phi(t, z)(M_0 - M_r) \sum_{s=a}^{t-1} R(\Phi)^*(s, \bar{z})W(s)R(g)(s), \\ T_{r2}(\tilde{g})(t) &:= \Phi(t, z)(N_0 - N_r) \sum_{s=t}^{b_r} R(\Phi)^*(s, \bar{z})W(s)R(g)(s), \\ T_{r3}(\tilde{g})(t) &:= \Phi(t, z)N_0 \sum_{s=b_r+1}^{+\infty} R(\Phi)^*(s, \bar{z})W(s)R(g)(s). \end{aligned}$$

Consequently,

$$\delta_{r1}(\tilde{g}) \leq 3 \sum_{i=1}^3 \sum_{t \in \mathcal{I}_r} R(T_{ri}(\tilde{g}))^*(t)W(t)R(T_{ri}(\tilde{g}))(t). \quad (5.17)$$

Denote

$$\begin{aligned} m_0 &:= \|M_0\|_1, \quad n_0 := \|N_0\|_1, \quad m_r := \|M_0 - M_r\|_1, \quad n_r := \|N_0 - N_r\|_1, \\ \alpha_0(z) &:= \max_{1 \leq i \leq 2n} \{\|\phi_i(\cdot, z)\|\}, \quad \alpha_r(z) := \max_{1 \leq i \leq 2n} \left\{ \sum_{t=b_r+1}^{\infty} R(\phi_i)^*(t, z)W(t)R(\phi_i)(t, z) \right\}. \end{aligned} \quad (5.18)$$

Then $m_r \rightarrow 0$ and $n_r \rightarrow 0$ by Proposition 5.3, and $\alpha_r(z) \rightarrow 0$ as $r \rightarrow \infty$. For convenience, denote

$$h_r(t) := (M_0 - M_r) \sum_{s=a}^{t-1} R(\Phi)^*(s, \bar{z})W(s)R(g)(s),$$

then

$$R(T_{r1}(\tilde{g}))(t) = R(\Phi(t, z)h_r(t)) = R(\Phi(t, z))h_r(t) + \text{diag}\{I_n, 0\}\Phi(t+1, z)\Delta h_r(t), \quad (5.19)$$

$$\|h_r(t)\|_1 \leq \|M_0 - M_r\|_1 \left\| \sum_{s=a}^{t-1} R(\Phi)^*(s, \bar{z})W(s)R(g)(s) \right\|_1 \leq \sqrt{2nm_r}\alpha_0(\bar{z})\|\tilde{g}\|. \quad (5.20)$$

In addition, it follows from (1.1_z) that

$$\Phi(t+1, z) = \begin{pmatrix} I_n & 0 \\ C(t) - zW_1(t) & I_n - A^*(t) \end{pmatrix} R(\Phi)(t, z).$$

Inserting it into (5.19), we get that

$$R(T_{r1}(\tilde{g}))(t) = R(\Phi(t, z))h_r(t) + \text{diag}\{I_n, 0\}R(\Phi)(t, z)\Delta h_r(t).$$

Therefore,

$$\begin{aligned} & R(T_{r1}(\tilde{g}))^*(t)W(t)R(T_{r1}(\tilde{g}))(t) \\ &= h_r^*(t)R(\Phi)^*(t, z)W(t)R(\Phi)(t, z)h_r(t) \\ & \quad + h_r^*(t)R(\Phi)^*(t, z)\text{diag}\{W_1(t), 0\}R(\Phi)(t, z)\Delta h_r(t) \\ & \quad + \Delta h_r^*(t)R(\Phi)^*(t, z)\text{diag}\{W_1(t), 0\}R(\Phi)(t, z)h_r(t) \\ & \quad + \Delta h_r^*(t)R(\Phi)^*(t, z)\text{diag}\{W_1(t), 0\}R(\Phi)(t, z)\Delta h_r(t). \end{aligned} \tag{5.21}$$

Since \mathcal{L} is in l.c.c. at $t = +\infty$, all the solutions of (1.1_z) are in $\mathcal{L}_W^2(\mathcal{I})$, and so $\Phi(\cdot, z) \in \mathcal{L}_W^2(\mathcal{I})$. It follows that all the diagonal entries of $R(\Phi)^*(t, z)W(t)R(\Phi)(t, z)$ are nonnegative and absolutely summable over $[a, +\infty)$. In addition, using the nonnegativity of $W(t)$, one has that

$$\sum_{t=a}^{+\infty} \|R(\Phi)^*(t, z)W(t)R(\Phi)(t, z)\|_1 \leq 2n\alpha_0^2(z), \tag{5.22}$$

which implies that

$$\sum_{t=a}^{+\infty} \|R(\Phi)^*(t, z)\text{diag}\{W_1(t), 0\}R(\Phi)(t, z)\|_1 \leq 2n\alpha_0^2(z),$$

which, together with (5.20)-(5.22), yields that

$$\begin{aligned} & \sum_{t \in \mathcal{I}_r} R(T_{r1}(\tilde{g}))^*(t)W(t)R(T_{r1}(\tilde{g}))(t) \\ & \leq \sum_{t \in \mathcal{I}_r} \|R(\Phi)^*(t, z)W(t)R(\Phi)(t, z)\|_1 [\|h_r(t)\|_1^2 + 2\|h_r(t)\|_1 \|\Delta h_r(t)\|_1 + \|\Delta h_r(t)\|_1^2] \\ & \leq 36n^2\alpha_0^2(z)\alpha_0^2(\bar{z})m_r^2\|\tilde{g}\|^2. \end{aligned} \tag{5.23}$$

Similarly, we get that

$$\begin{aligned} & \sum_{t \in \mathcal{I}_r} R(T_{r2}(\tilde{g}))^*(t)W(t)R(T_{r2}(\tilde{g}))(t) \leq 36n^2\alpha_0^2(z)\alpha_0^2(\bar{z})n_r^2\|\tilde{g}\|^2, \\ & \sum_{t \in \mathcal{I}_r} R(T_{r3}(\tilde{g}))^*(t)W(t)R(T_{r3}(\tilde{g}))(t) \leq 4n^2\alpha_0^2(z)\alpha_r(\bar{z})n_0^2\|\tilde{g}\|^2. \end{aligned}$$

Thus, from (5.17), it follows that

$$\delta_{r1}(\tilde{g}) \leq 12n^2\alpha_0^2(z) \left[9\alpha_0^2(\bar{z})(m_r^2 + n_r^2) + \alpha_r(\bar{z})n_0^2 \right] \|\tilde{g}\|^2. \tag{5.24}$$

With a similar argument to that used for $\delta_{r1}(\tilde{g})$, one can show that

$$\delta_{r2}(\tilde{g}) \leq 72n^2(m_0^2 + n_0^2)\alpha_0^2(\bar{z})\alpha_r(z)\|\tilde{g}\|^2,$$

which, together with (5.24), yields that

$$\delta_r(\tilde{g}) = \delta_{r1}(\tilde{g}) + \delta_{r2}(\tilde{g}) \leq \eta(r)\|\tilde{g}\|^2, \tag{5.25}$$

where

$$\eta(r) = 12n^2 \left[9\alpha_0^2(z)\alpha_0^2(\bar{z})(m_r^2 + n_r^2) + n_0^2\alpha_0^2(z)\alpha_r(\bar{z}) + 6(m_0^2 + n_0^2)\alpha_0^2(\bar{z})\alpha_r(z) \right]. \quad (5.26)$$

It is obvious that $\eta(r) \rightarrow 0$ as $r \rightarrow \infty$, which implies that assertion (i) holds.

It can be directly derived from Lemma 4.1, Theorem 5.1, assertion (i), and Lemma 2.10 that $\{H_{1,r}\}$ is spectrally exact for H_1 by the assumption that $0 \notin \sigma(H_1)$. The whole proof is complete.

In order to further study how to approximate the spectrum of H_1 by those of $\{H_{1,r}\}$, we first give the following useful result:

Theorem 5.3. *Every self-adjoint subspace extension H_1 of H_0 has a pure discrete spectrum in the case that \mathcal{L} is in l.c.c. at $t = +\infty$.*

Proof. According to [42, Theorems 6.7 and 6.10] and Lemma 2.1, it suffices to prove that $(zI - H_1)^{-1}$ is a Hilbert-Schmidt operator for any $z \in \rho(H_1)$.

By Proposition 5.1, for any $z \in \rho(H_1)$ and any $\tilde{g} \in L_W^2(\mathcal{I})$,

$$(zI - H_1)^{-1}(\tilde{g})(t) = \sum_{s=a}^{+\infty} G(t, s, z)W(s)R(g)(s), \quad t \in \mathcal{I},$$

where $G(t, s, z)$ is given by (5.7). Define

$$\begin{aligned} \mathcal{F}_1(\tilde{g})(t) &:= \sum_{s=a}^{+\infty} F_1(t, s, z)W(s)R(g)(s), \quad t \in \mathcal{I}, \\ \mathcal{F}_2(\tilde{g})(t) &:= \sum_{s=a}^{+\infty} F_2(t, s, z)W(s)R(g)(s), \quad t \in \mathcal{I}, \end{aligned}$$

where

$$\begin{aligned} F_1(t, s, z) &= \begin{cases} \Phi(t, z)M_0R(\Phi)^*(s, \bar{z}), & a \leq s < t < +\infty, \\ 0, & a \leq t \leq s < +\infty, \end{cases} \\ F_2(t, s, z) &= \begin{cases} 0, & a \leq s < t < +\infty, \\ \Phi(t, z)N_0R(\Phi)^*(s, \bar{z}), & a \leq t \leq s < +\infty. \end{cases} \end{aligned}$$

Obviously, $(zI - H_1)^{-1} = \mathcal{F}_1 + \mathcal{F}_2$. Therefore, it is sufficient to prove that \mathcal{F}_1 and \mathcal{F}_2 are both Hilbert-Schmidt operators. Denote $N := \dim L_W^2(\mathcal{I})$. In the case of $N < \infty$, \mathcal{F}_1 and \mathcal{F}_2 are obviously Hilbert-Schmidt operators. So, it is only needed to show that this assertion holds in the case of $N = \infty$. We first prove that \mathcal{F}_1 is a Hilbert-Schmidt operator in this case. Let $\{\tilde{e}_j : j \in \mathbf{Z}^+\}$ be an orthonormal basis of $L_W^2(\mathcal{I})$. Then

$$\mathcal{F}_1(\tilde{e}_j)(t) = \sum_{s=a}^{+\infty} F_1(t, s, z)W(s)R(e_j)(s) = \Phi(t, z)u_j(t), \quad t \in \mathcal{I},$$

where

$$u_j(t) := M_0 \sum_{s=a}^{t-1} R(\Phi)^*(s, \bar{z}) W(s) R(e_j)(s).$$

Then

$$\|u_j(t)\|_1 \leq \|M_0\|_1 \left\| \sum_{s=a}^{t-1} R(\Phi)^*(s, \bar{z}) W(s) R(e_j)(s) \right\|_1 \leq m_0 \left(\sum_{i=1}^{2n} |\langle \phi_i(\cdot, \bar{z}), e_j \rangle|^2 \right)^{1/2}. \quad (5.27)$$

Similar to the discussions for (5.21) and (5.23) with replacing $h_r(t)$ by $u_j(t)$, one has that

$$\begin{aligned} \sum_{j=1}^{\infty} \|\mathcal{F}_1(\tilde{e}_j)\|^2 &= \sum_{j=1}^{\infty} \sum_{t=a}^{+\infty} R(\Phi(t, z) u_j(t))^* W(t) R(\Phi(t, z) u_j(t)) \\ &\leq 18 n m_0^2 \alpha_0^2(z) \sum_{i=1}^{2n} \sum_{j=1}^{\infty} |\langle \phi_i(\cdot, \bar{z}), e_j \rangle|^2 \\ &\leq 36 n^2 m_0^2 \alpha_0^2(z) \alpha_0^2(\bar{z}) < \infty, \end{aligned}$$

in which (5.27), (5.22), and Parseval's identity have been used. Therefore, \mathcal{F}_1 is a Hilbert-Schmidt operator. Similarly, one can show that \mathcal{F}_2 is a Hilbert-Schmidt operator and thus $(zI - H_1)^{-1}$ is a Hilbert-Schmidt operator. The proof is complete.

Remark 5.1. With a similar argument, by applying the Green function of $H_{1,r}$ given in Proposition 5.2, it can be easily verified that the resolvent of $H_{1,r}$ is a Hilbert-Schmidt operator. Hence, the resolvent of $\hat{H}_{1,r}$ is also a Hilbert-Schmidt operator by (4.2). In addition, denote $\sigma(H_1) \setminus \{0\} := \{\lambda_k : k \in \Upsilon\}$, where Υ denotes the eigenvalue index set of H_1 . Then we can further get that $\sum_{k \in \Upsilon} |\lambda_k|^{-2} < \infty$ by the fact that the resolvent of H_1 is a Hilbert-Schmidt operator.

By Theorem 5.3, H_1 has a discrete spectrum in the case that \mathcal{L} is in l.c.c. at $t = +\infty$. By translating it if necessary, we may suppose that 0 is not an eigenvalue of H_1 . The eigenvalues of H_1 may be ordered as (multiplicity included):

$$\cdots \leq \lambda_{-2} \leq \lambda_{-1} < 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots.$$

For convenience, by Υ denote the eigenvalue index set of H_1 and $\sigma(H_1) = \{\lambda_k : k \in \Upsilon\}$. By (ii) of Theorem 5.2, $\{H_{1,r}\}$ is spectrally exact for H_1 if $0 \notin \sigma(H_1)$. Hence, since $0 \notin \sigma(H_1)$, there exists r_0 such that $0 \notin \sigma(H_{1,r})$ for all $r \geq r_0$. Therefore, for $r \geq r_0$, the eigenvalues of $H_{1,r}$ may be ordered as (multiplicity included):

$$\lambda_{-m(r)}^{(r)} \leq \cdots \leq \lambda_{-2}^{(r)} \leq \lambda_{-1}^{(r)} < 0 < \lambda_1^{(r)} \leq \lambda_2^{(r)} \leq \cdots \leq \lambda_{n(r)}^{(r)},$$

where $m(r)$ and $n(r)$ are the numbers of negative and positive eigenvalues of $H_{1,r}$, respectively. For convenience, we briefly denote the eigenvalue index set of $H_{1,r}$ by Υ_r , and then

$\sigma(H_{1,r}) = \{\lambda_k^{(r)} : k \in \Upsilon_r\}$. By Lemma 4.1, $\sigma(H_{1,r}) = \sigma(\hat{H}_{1,r})$, which implies that $0 \in \rho(\hat{H}_{1,r})$ as $r \geq r_0$.

Theorem 5.4. *Assume that \mathcal{L} is in l.c.c. at $t = +\infty$. For each $k \in \Upsilon$, there exists an $r_k \geq r_0$ such that $k \in \Upsilon_r$ for $r \geq r_k$, and $\lambda_k^{(r)} \rightarrow \lambda_k$ as $r \rightarrow \infty$.*

Proof. Let

$$S = (-H_1)^{-1}, \quad S_r = (-\hat{H}_{1,r})^{-1}, \quad r \geq r_0.$$

Then, according to Lemmas 2.3 and 4.1, the proof of Theorem 5.3, and Remark 5.1, it follows that $S_r P_r$ and S are both self-adjoint and Hilbert-Schmidt operators in $L_W^2(\mathcal{I})$. And $\mu_k = -1/\lambda_k$ for $k \in \Upsilon$ and $\mu_k^{(r)} = -1/\lambda_k^{(r)}$ for $k \in \Upsilon_r$ are eigenvalues of S and $S_r P_r$, respectively, by Lemmas 2.1 and 2.3. ($S_r P_r$ also has 0 as an eigenvalue of infinite multiplicity. But it is not related to $H_{1,r}$ or H_1 , and so can be ignored.) Further, $S_r P_r \rightarrow S$ in norm as $r \rightarrow \infty$ by (i) of Theorem 5.2. It follows that $S_r P_r \rightarrow S$ in the norm resolvent sense as $r \rightarrow \infty$ according to the proof of [23, Theorem 8.18] (for the concept of convergence of self-adjoint operators in the norm resolvent sense, please see [23, 42]). Let $E(S_r P_r, \lambda)$ and $E(S, \lambda)$ be spectral families of $S_r P_r$ and S , respectively. Then, by (b) of [23, Theorem 8.23] one has that for any $\alpha, \beta \in \mathbf{R} \cap \rho(S)$ with $\alpha < \beta$,

$$\| [E(S_r P_r, \beta) - E(S_r P_r, \alpha)] - [E(S, \beta) - E(S, \alpha)] \| \rightarrow 0 \text{ as } r \rightarrow \infty,$$

which, together with [42, Theorem 4.35], yields that

$$\dim R[E(S_r P_r, \beta) - E(S_r P_r, \alpha)] = \dim R[E(S, \beta) - E(S, \alpha)]$$

for all sufficiently large r . Hence, for each $k \in \Upsilon$, there exists an $r_k \geq r_0$ such that $k \in \Upsilon_r$ for $r \geq r_k$.

Next, we show that $\lambda_k^{(r)} \rightarrow \lambda_k$ as $r \rightarrow \infty$. To do so, it suffices to prove that $\mu_k^{(r)} \rightarrow \mu_k$ as $r \rightarrow \infty$. The eigenvalues are described by the Courant-Fischer min-max theorem in the case of $\dim L_W^2(\mathcal{I}) < \infty$ and by a min-max principle according to [26, Section 12.1] in the case of $\dim L_W^2(\mathcal{I}) = \infty$, respectively; that is,

$$\mu_k = \begin{cases} \min_{V_k} \max_{\substack{x \in V_k, \\ \|x\|=1}} \langle Sx, x \rangle, & k \in \Upsilon \text{ with } k > 0, \\ \max_{V_k} \min_{\substack{x \in V_k, \\ \|x\|=1}} \langle Sx, x \rangle, & k \in \Upsilon \text{ with } k < 0, \end{cases} \quad (5.28)$$

where V_k runs through all the $|k|$ -dimensional subspaces of $L_W^2(\mathcal{I})$. For $r \geq r_k$, $\mu_k^{(r)}$ is similarly expressed in terms of $\langle S_r P_r x, x \rangle$; that is,

$$\mu_k^{(r)} = \begin{cases} \min_{V_k} \max_{\substack{x \in V_k, \\ \|x\|=1}} \langle S_r P_r x, x \rangle, & k \in \Upsilon \text{ with } k > 0, \\ \max_{V_k} \min_{\substack{x \in V_k, \\ \|x\|=1}} \langle S_r P_r x, x \rangle, & k \in \Upsilon \text{ with } k < 0. \end{cases} \quad (5.29)$$

We first consider the case that $k \in \Upsilon$ with $k > 0$. Let $r \geq r_k$. It follows from (5.28)-(5.29) that there exist two k -dimensional subspaces V_k and \tilde{V}_k of $L_W^2(\mathcal{I})$ such that

$$\mu_k = \max_{\substack{x \in V_k, \\ \|x\|=1}} \langle Sx, x \rangle, \quad \mu_k^{(r)} = \max_{\substack{x \in \tilde{V}_k, \\ \|x\|=1}} \langle S_r P_r x, x \rangle. \quad (5.30)$$

In addition, there exist $x_1 \in \tilde{V}_k$ with $\|x_1\| = 1$ and $x_2 \in V_k$ with $\|x_2\| = 1$ such that

$$\max_{\substack{x \in \tilde{V}_k, \\ \|x\|=1}} \langle Sx, x \rangle = \langle Sx_1, x_1 \rangle, \quad \max_{\substack{x \in V_k, \\ \|x\|=1}} \langle S_r P_r x, x \rangle = \langle S_r P_r x_2, x_2 \rangle. \quad (5.31)$$

From (5.28)-(5.31), we have

$$\begin{aligned} \mu_k - \mu_k^{(r)} &\leq \max_{\substack{x \in \tilde{V}_k, \\ \|x\|=1}} \langle Sx, x \rangle - \max_{\substack{x \in \tilde{V}_k, \\ \|x\|=1}} \langle S_r P_r x, x \rangle \leq \langle (S - S_r P_r)x_1, x_1 \rangle, \\ \mu_k - \mu_k^{(r)} &\geq \max_{\substack{x \in V_k, \\ \|x\|=1}} \langle Sx, x \rangle - \max_{\substack{x \in V_k, \\ \|x\|=1}} \langle S_r P_r x, x \rangle \geq \langle (S - S_r P_r)x_2, x_2 \rangle. \end{aligned}$$

Therefore, it follows that

$$\begin{aligned} |\mu_k - \mu_k^{(r)}| &\leq \max\{|\langle (S - S_r P_r)x_1, x_1 \rangle|, |\langle (S - S_r P_r)x_2, x_2 \rangle|\} \\ &\leq \|S - S_r P_r\| \rightarrow 0 \text{ as } r \rightarrow \infty. \end{aligned} \quad (5.32)$$

Thus, $\mu_k^{(r)} \rightarrow \mu_k$ as $r \rightarrow \infty$ for $k \in \Upsilon$ with $k > 0$.

Similarly, one can get that $\mu_k^{(r)} \rightarrow \mu_k$ as $r \rightarrow \infty$ for $k \in \Upsilon$ with $k < 0$. This completes the proof.

At the end of this section, we shall try to give an error estimate for the approximation of λ_k by $\lambda_k^{(r)}$ for each $k \in \Upsilon$. Obviously, it is very important in numerical analysis and applications. In order to give error estimates of $\lambda_k^{(r)}$ to λ_k , in view of $\lambda_k = -1/\mu_k$ and $\lambda_k^{(r)} = -1/\mu_k^{(r)}$, we shall first investigate the error estimates of $\mu_k^{(r)}$ to μ_k for $k \in \Upsilon$ instead.

In view of the arbitrariness of $\lambda \in \mathbf{R}$ in (2.4), we might as well take $\lambda = 0$ in (2.4) in the rest of this section.

Proposition 5.4. *Assume that \mathcal{L} is in l.c.c. at $t = +\infty$. Then, for each $k \in \Upsilon$ and $r \geq r_k$, where r_k is specified in Theorem 5.4,*

$$|\mu_k^{(r)} - \mu_k| \leq 2\sqrt{3}n\alpha_0[(6m_0^2 + 7n_0^2)(\|E(a)\|_1^2 + \|E(a)B(a)\|_1^2 + n)]^{1/2}\varepsilon_r^{1/2}, \quad (5.33)$$

where $\alpha_0 := \alpha_0(0)$, m_0 , and n_0 are constants and given by (5.18), $E(a) = (I_n - A(a))^{-1}$, and ε_r is completely determined by the coefficients of (1.1), more precisely, it is determined by (5.36), (5.38), (5.40), and (5.42). In addition, $\varepsilon_r \rightarrow 0$ as $r \rightarrow \infty$.

Proof. In view of $0 \in \rho(H_1) \cap \rho(\hat{H}_{1,r})$ as $r \geq r_0$, it follows from (5.26) with $z = 0$ and (5.32) that

$$|\mu_k^{(r)} - \mu_k| \leq 2\sqrt{3}n\alpha_0[9\alpha_0^2(m_r^2 + n_r^2) + (6m_0^2 + 7n_0^2)\alpha_r]^{1/2}, \quad r \geq r_0, \quad (5.34)$$

where $\alpha_r := \alpha_r(0)$ and $\alpha_0 := \alpha_0(0)$, and $m_r, n_r, m_0, n_0, \alpha_r(z), \alpha_0(z)$ are specified in (5.18).

By the arbitrariness of $\lambda \in \mathbf{R}$ in (2.4), we take $\lambda = 0$ in it. So it follows from (3.4) that $\omega_1, \dots, \omega_{2n}$ are solutions of (1.1 $_\lambda$) with $\lambda = 0$ in $[t_0 + 1, +\infty)$, where t_0 is specified by (\mathbf{A}_2). In addition, since $\phi_1(\cdot, 0), \dots, \phi_{2n}(\cdot, 0)$ are solutions of (1.1 $_z$) with $z = 0$ in \mathcal{I} and $b_r > t_0$, by (2.2), (5.11), and (5.15) we get that $K = K_r$, which, together with (5.12), yields that $M_r = M_0$, $N_r = N_0$, and thus $m_r = n_r = 0$ by (5.18).

Now, it remains to estimate α_r . By (\mathbf{A}_1), we get that every solution y of (1.1 $_z$) with $z = 0$ satisfies

$$R(y)(t+1) = U(t)R(y)(t), \quad t \geq a, \quad (5.35)$$

where $E(t) = (I_n - A(t))^{-1}$,

$$U(t) = \begin{pmatrix} E(t+1) + E(t+1)B(t+1)C(t) & E(t+1)B(t+1)(I_n - A^*(t)) \\ C(t) & I_n - A^*(t) \end{pmatrix}. \quad (5.36)$$

It follows that

$$\begin{aligned} & R(y)^*(t+1)W(t+1)R(y)(t+1) \\ &= R(y)^*(t)U^*(t)W(t+1)U(t)R(y)(t) \\ &= R(y)^*(t-1)U^*(t-1)U^*(t)W(t+1)U(t)U(t-1)R(y)(t-1) \\ &= R(y)^*(a)V^*(t)W(t+1)V(t)R(y)(a), \quad t \geq a, \end{aligned} \quad (5.37)$$

where

$$V(t) := U(t)U(t-1) \cdots U(a), \quad t \geq a. \quad (5.38)$$

Since \mathcal{L} is in l.c.c. at $t = +\infty$, it follows from (5.37) that

$$\begin{aligned} & \sum_{t=b_r}^{\infty} R(y)^*(t+1)W(t+1)R(y)(t+1) \\ &= R(y)^*(a)D_r R(y)(a) \rightarrow 0 \text{ as } r \rightarrow \infty, \end{aligned} \quad (5.39)$$

where

$$D_r := \sum_{t=b_r}^{\infty} V^*(t)W(t+1)V(t). \quad (5.40)$$

Then, D_r is positive semi-definite since $W(t)$ is positive semi-definite for $t \geq a$. In addition, since

$$R(y)(a) = \begin{pmatrix} E(a) & E(a)B(a) \\ 0 & I_n \end{pmatrix} y(a), \quad (5.41)$$

$R(y)(a)$ can be taken any complex vector belonging to \mathbf{C}^{2n} . Denote

$$\varepsilon_r := \|D_r\|_1. \quad (5.42)$$

Then, combining the positive semi-definiteness of D_r and the arbitrariness of $R(y)(a)$, it follows from (5.39) that $\varepsilon_r \rightarrow 0$ as $r \rightarrow \infty$. In addition, since $\phi_1(\cdot, 0), \dots, \phi_{2n}(\cdot, 0)$ satisfy (5.35)-(5.41), it follows from (5.39) and (5.41) that

$$\begin{aligned}
\alpha_r &= \max_{1 \leq i \leq 2n} \{R(\phi_i)^*(a, 0)D_r R(\phi_i)(a, 0)\} \\
&\leq \varepsilon_r \max_{1 \leq i \leq 2n} \{\|R(\phi_i)(a, 0)\|_1^2\} \\
&\leq \varepsilon_r (\|E(a)\|_1^2 + \|E(a)B(a)\|_1^2 + n),
\end{aligned} \tag{5.43}$$

in which $\Phi(a, 0) = (\phi_1, \dots, \phi_{2n})(a, 0) = I_{2n}$ have been used. Inserting it and $m_r = n_r = 0$ into (5.34), we get that (5.33) holds. The proof is complete.

Theorem 5.5. *Assume that \mathcal{L} is in l.c.c. at $t = +\infty$. Then, for each $k \in \Upsilon$, there exists an $r'_k \geq r_k$, where r_k is specified in Theorem 5.4, such that for all $r \geq r'_k$,*

$$|\lambda_k^{(r)} - \lambda_k| \leq \frac{|\lambda_k^{(r)}|^2 e_r}{1 - |\lambda_k^{(r)}| e_r}, \tag{5.44}$$

$$|\lambda_k^{(r)} - \lambda_k| \leq \frac{|\lambda_k|^2 e_r}{1 - |\lambda_k| e_r}, \tag{5.45}$$

where e_r denotes the number on the right-hand side in (5.33).

Proof. For each $k \in \Upsilon$, λ_k and $\lambda_k^{(r)}$ have the same sign for sufficiently large r . In view of $\lambda_k = -1/\mu_k$ and $\lambda_k^{(r)} = -1/\mu_k^{(r)}$, it follows from (5.33) that for each $k \in \Upsilon$,

$$\left| \frac{1}{\lambda_k^{(r)}} - \frac{1}{\lambda_k} \right| \leq e_r, \quad r \geq r_k,$$

which yields that

$$|\lambda_k^{(r)} - \lambda_k| \leq e_r |\lambda_k^{(r)}| |\lambda_k|, \quad r \geq r_k. \tag{5.46}$$

Thus,

$$|\lambda_k| = |\lambda_k + \lambda_k^{(r)} - \lambda_k^{(r)}| \leq |\lambda_k - \lambda_k^{(r)}| + |\lambda_k^{(r)}| \leq e_r |\lambda_k^{(r)}| |\lambda_k| + |\lambda_k^{(r)}|, \quad r \geq r_k,$$

which implies that

$$|\lambda_k| (1 - |\lambda_k^{(r)}| e_r) \leq |\lambda_k^{(r)}|. \tag{5.47}$$

By Theorem 5.4 and Proposition 5.4, there exists an $r'_k \geq r_k$ such that $1 - |\lambda_k^{(r)}| e_r > 0$. Hence, it follows from (5.46) and (5.47) that (5.44) holds. With a similar argument, one can show that (5.45) holds. This completes the proof.

6 Spectral approximation in the limit point and intermediate cases

Now, we study the regular approximation of spectra of (1.1) in the case that \mathcal{L} is either in l.p.c. or the intermediate case at $t = +\infty$, namely, $n \leq d < 2n$. In each case, we show that $\{H_{1,r}\}_{r=1}^\infty$ is spectrally inclusive for any given self-adjoint subspace extension H_1 . We always

assume that $(\mathbf{A}_1) - (\mathbf{A}_3)$ hold when \mathcal{L} is in l.p.c. at $t = +\infty$ and $(\mathbf{A}_1) - (\mathbf{A}_4)$ hold when \mathcal{L} is in the intermediate case at $t = +\infty$.

Theorem 6.1. *Assume that $n \leq d < 2n$. Let H_1 be any fixed SSE of H_0 , and $H_{1,r}$ the induced regular SSE of H_1 on \mathcal{I}_r , where H_1 and $H_{1,r}$ are determined by (3.9) and (3.12), respectively, when $d = n$ (l.p.c.), and they are determined by (3.16) and (3.17), respectively, when $n < d < 2n$ (the intermediate case). And let $H'_{1,r}$ be defined by (4.9). Then*

- (i) $\{H'_{1,r}\}$ is SRC to H_1 ;
- (ii) $\{H_{1,r}\}$ is spectrally inclusive for H_1 if $0 \notin \sigma(H_1)$.

Proof. The main idea of the proof is similar to that of Theorem 5.1, where the core $C(H_1)$ of H_1 in (5.3) is replaced by

$$C(H_1) = H_{00} \dot{+} L\{\beta_1, \dots, \beta_d\},$$

where $\{\beta_i = (\omega_i, \tilde{\tau}_i)\}_{i=1}^d$ is a GKN-set for $\{H_0, H_0^*\}$ and $\omega_i, 1 \leq i \leq d$, is defined by (3.8) and (3.15) when $d = n$ (l.p.c.) and $n < d < 2n$ (the intermediate case), separately. So its details are omitted. The proof is complete.

Remark 6.1. In the case that \mathcal{L} is in l.p.c. at $t = +\infty$, the sequence of induced regular self-adjoint subspace extensions $\{H_{1,r}\}$ is spectrally inclusive for H_1 , but not spectrally exact for H_1 in general. For a counterexample, the reader is referred to [19, Example 3.1].

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